Topological rigidity of quoric manifolds

Ioannis Gkeneralis
(University of the Aegean/Department of Mathematics, Samos, Greece)
E-mail: igkeneralis@math.aegean.gr

The basic problem in Geometric Topology is the topological classification of manifolds, spaces that are locally like the usual Euclidean spaces, like the surfaces. More precisely, we study manifolds that have the same algebraic properties (homotopy equivalences) and we would like to show that they are equivalent (homeomorphic). There are a lot of conjectures towards this direction with the strongest being the Isomorphism Conjecture of Farrell-Jones. Furthermore, there are the corresponding conjectures when the manifolds are equipped with a group of symmetries (group actions). In this case, all the structures (homotopy equivalences, homeomorphisms) should preserve the group action (equivariant).

The original idea of the classification problems is Mostow's Rigidity Theorem in which it was proved that two hyperbolic manifolds, of dimension larger than 2, which are homotopy equivalent, they are isometric. This result is the basis of most of the conjectures of classification and rigidity. Usually, one of the two manifolds has nice properties (nonpositive curvature, hyperbolic fundamental group) and the other is simply homotopy equivalent to the first. The problem is to equip the second manifold with the properties of the first through the homotopy equivalence. After that, geometric methods, similar to the one in Mostow's Theorem, will give the result.

In the case of interest, we start with Euclidean spaces $\mathbb{R}^n$ on which we can define a multiplication such that, if $x, y \neq 0$, then $xy \neq 0$. If we insist that the multiplication is associative, then $n = 1, 2, 4$ from Frobenious Theorem. In the first case we have the multiplication of the reals, in the second case we have the complex multiplication and in the third case, we have the multiplication on the quaternions which is not commutative. The corresponding spheres, in each case, are the elements of length one and for $n = 1$ is the group of two elements $\mathbb{Z}_2$, for $n = 2$ is the unit circle S^1, for $n = 4$ is the unit sphere in $\mathbb{R}^4$, S^3, and they are all groups with the induced multiplication. In each of these dimension we have the corresponding torus, $Z^n = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, $T^n = S^1 \times \ldots \times S^1$ and $Q^n = S^3 \times \ldots \times S^3$. In each case, starting with a polyhedron, we can construct a space on which the tori act and the quotient space in the original polyhedron. These manifolds are called standard models. The local action is given in fact as the corresponding multiplication.

In each case, we start with a manifold on which the tori act (locally linearly) and they are homotopy equivalent to the standard model, preserving the group action and we want to show that is homeomorphic to the standard model. In all case, the process is similar. Let N be the manifold that we study. First, we show that the action has the local properties of the standard model. Thus the quotient space is a polyhedron P. Next, we construct the standard model from P. The final result is consequence of two results: first that N is homeomorphic to the canonical model over P, which is homeomorphic to the original standard model.

For $n = 1$, we have a much richer structure than those of finite groups. In this case, we have actions of groups that are generated by reflections (Coxeter groups). The basic properties are given in [2]. The rigidity theorem is proved in [6]. In this case, we have to show that the elements that act as reflections in the standard model, act as reflections on N.

For $n = 2$ we have the toric manifolds, which are the non-singular toric varieties and their topological analogue, the quasitoric manifolds ([1], [3]). The result is given in [5]. To show that the action of T^n on N is locally standard, we study the representation s of T^n. To show that N is homeomorphic to the standard model of the action, we show that an element in the local Čech cohomology of the quotient map vanishes.

For the remaining case, we work along the lines of the Coxeter groups and quasitoric varieties. Let $Q^n = (S^3)^n$. We say that Q^n acts on a manifold M^{4n} locally regularly if, locally, the action is given by (quaternionic) multiplication or conjugation on each coordinate. Then the quotient is a manifold with corners. Conversely, starting with a manifold with corners and an appropriate function from its faces to the conjugacy classes of subgroups of Q^n, we can construct a locally regular (quoric) manifold. Our main result is the following.

Theorem 1. *(Rigidity of Quoric Manifolds). Let M^{4n} be a closed locally regular quoric manifold over a nice n-manifold with corners X and X is a homotopy polytope i.e. all the faces of X (and X itself) are contractible manifolds with corners. Let N^{4n} be a locally linear closed Q^n-manifold and $f : N^{4n} \longrightarrow M^{4n}$ a Q^n-equivariant homotopy equivalence. Then f is Q^n-homotopic to a Q^n-homeomorphism.*

The proof of the main theorem follows the methods of [6] and [5].

- We show that the action on N^n is locally regular. For this result, we first prove that N^n has the same isotropy groups as M^n and f is an isovariant homotopy equivalence. Then we prove that the action on N^n is locally regular. That is quite different from the torus case. The reason is that this part depends on the representation theory of the underlying group. But Q^n is not abelian and thus its representation theory is more complicated that that of T^n. So, we need a more thorough analysis in this case.
- Let Y be the quotient manifold with corners of the action. We prove that N^n is Q^n-homeomorphic with the standard model constructed from Y. For this, there is an obstruction theory analogous to the torus case.
- The rest is standard. The map f induces a face preserving homotopy equivalence $\phi : Y \longrightarrow X$. Induction and standard surgery methods imply that ϕ is face homotopic to a face homeomorphism ψ. The map ψ induces a Q^n-homeomorphism $g : N^n \longrightarrow M^n$ that is homotopic to f.

References

[1] Buchstaber, Victor M.; Panov, Taras E. *Torus actions and their applications in topology and combinatorics.* University Lecture Series, 24. American Mathematical Society, Providence, RI, 2002.

[2] Davis, Michael W. *Groups generated by reflections and aspherical manifolds not covered by Euclidean space.* Ann. of Math. (2) 117 (1983), no. 2, 293?324.

[3] Davis, Michael W.; Januszkiewicz, Tadeusz *Convex polytopes, Coxeter orbifolds and torus actions.* Duke Math. J. 62 (1991), no. 2, 417?451.

[4] Hopkinson, Jeremy *Quoric Manifolds.* PhD Thesis, University of Manchester, School of Mathematics, 2012.

[5] Metaftsis, Vassilis; Prassidis, Stratos, *Topological rigidity of quasitoric manifolds.* Math. Scand. 122 (2018), no. 2, 179-196.

[6] Prassidis, Stratos; Spieler, Barry, *Rigidity of Coxeter groups.* Trans. Amer. Math. Soc. 352 (2000), no. 6, 2619-2642.