Octonionic Stiefel manifolds and vector bundles

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The octonions $\mathbb{O}$ satisfy a weaker form of associativity. Namely, they are alternative and power associative only and are not as well known as complex numbers $\mathbb{C}$ and the quaternions $\mathbb{H}$ which are much more widely studied and used.

This talk studies: Stiefel and Grassmann varieties, and vector bundles over octonions $\mathbb{O}$.

Stiefel manifolds. Let $\mathbb{F}$ stand for $\mathbb{R}$, the reals, $\mathbb{C}$, the complex numbers or $\mathbb{H}$, the quaternions.

Recall that the Stiefel manifold $V_{n,r}(\mathbb{F})$ for $r \leq n$ is the set of all orthonormal r-frames in $\mathbb{F}^n$ which can be thought of as a set of $n \times k$ matrices by writing a r-frame as a matrix of k column vectors in $\mathbb{F}^n$. We then have

$$V_{n,r}(\mathbb{F}) = \left\{A \in M_{n,r}(\mathbb{F}) :\ \bar{A}^t A = I_r\right\}$$

and define

$$V_{n,r}(\mathbb{O}) = \left\{A \in M_{n,r}(\mathbb{O}) :\ \bar{A}^t A = I_r\right\}.$$

Those yield $V_{n,r}(\mathbb{F})$ and $V_{n,r}(\mathbb{O})$ as algebraic varieties over $\mathbb{R}$ (see [1, 3] for details).

Each $V_{n,r}(\mathbb{F})$ can be viewed as a homogeneous space:

$$V_{n,r}(\mathbb{F}) \cong \mathrm{U}(n,\mathbb{F})/\mathrm{U}(n-r,\mathbb{F}).$$

But, for $V_{n,r}(\mathbb{O})$ we have:

Proposition 1. (1) *$V_{n,r}(\mathbb{O})$ is a compact smooth submanifold of $M_{n,r}(\mathbb{O})$ for any $r \leq n$.*

(2) *$V_{n,r}(\mathbb{O})$ is path-connected for any $r \leq n$ and the map $\pi : V_{n,r+1}(\mathbb{O}) \to V_{n,r}(\mathbb{O})$, given by $\pi(A|v) = A$, is a smooth fibre bundle.*

Grassmann manifolds. Grassmann manifold $G_{n,r}(\mathbb{F})$ is a differentiable manifold that parameterizes the set of all r-dimensional linear subspaces of $\mathbb{F}^n$. Since the rank of an orthogonal projection operator equals its trace, we can identify

$$G_{n,r}(\mathbb{F}) = \left\{A \in M_n(\mathbb{F}) :\ A = \bar{A}^t = A^2,\ \mathrm{tr}(A) = r\right\}$$

and define

$$G_{n,r}(\mathbb{O}) = \left\{A \in M_n(\mathbb{O}) :\ A = \bar{A}^t = A^2,\ \mathrm{tr}(A) = r\right\}.$$

Those yield $G_{n,r}(\mathbb{F})$ and $G_{n,r}(\mathbb{O})$ as algebraic varieties over $\mathbb{R}$ (see [1, 3] for details).

Each $G_{n,r}(\mathbb{F})$ can be viewed as a homogeneous space:

$$G_{n,r}(\mathbb{F}) \cong \mathrm{U}(n,\mathbb{F})/\mathrm{U}(r,\mathbb{F}) \times \mathrm{U}(n-r,\mathbb{F}).$$

Furthermore, we have the principal $\mathrm{U}(r)$-bundle

$$\mathrm{U}(r,\mathbb{F}) \hookrightarrow V_{n,r}(\mathbb{F}) \to G_{n,r}(\mathbb{F})$$

for the Stiefel map $V_{n,r}(\mathbb{F}) \to G_{n,r}(\mathbb{F})$.

Due to the non-associativity of $\mathbb{O}$ we do not have a Stiefel map $\pi : V_{n,r}(\mathbb{O}) \to G_{n,r}(\mathbb{O})$, but we may define a subset $V'_{n,r}(\mathbb{O}) \subseteq V_{n,r}(\mathbb{O})$ as follows: $A \in V'_{n,r}(\mathbb{O})$ if the set of all entries of A generate an associative subalgebra of $\mathbb{O}$. Then, we have a Stiefel map $\pi : V'_{n,r}(\mathbb{O}) \to G_{n,r}(\mathbb{O})$ given by $\pi(A) = A\bar{A}^t$.

Similarly to $V'_{n,r}(\mathbb{O})$, we define $G'_{n,r}(\mathbb{O})$ as follows: $A \in G'_{n,r}(\mathbb{O})$ if all entries of A generate an associative subalgebra of $\mathbb{O}$. It is clear that the Stiefel map $\pi : V'_{n,r}(\mathbb{O}) \to G_{n,r}(\mathbb{O})$ yields a surjective map $\pi : V'_{n,r}(\mathbb{O}) \to G'_{n,r}(\mathbb{O})$. In particular, $G'_{n,r}(\mathbb{O})$ is piecewise-smooth path-connected.

Vector bundles

By analogy with real, complex or quaterionic vector bundles, define an octonionic vector bundle of rank r over some space X as a continuous map

$$f : X \to G'_{-,r}(\mathbb{O}).$$

The following result holds by adapting proof of [1, Theorem 12.1.7] or [2, Theorem 2.5]:

Theorem 2. *Any smooth map $X \to G'_{n,r}(\mathbb{O})$ with $r \leq n$ and X, a compact non-singular algebraic set, is homotopic to an entire rational map from X to $G'_{n,r}(\mathbb{O})$.*

Then, we derive:

Corollary 3. *Any smooth map $X \to G_{n,r}(\mathbb{O})$ with $n = 2, 3$ and $r = 1$ is homotopic to an entire rational map $X \to G_{n,r}(\mathbb{O})$. In particular, any continuous map $\mathbb{S}^n \to \mathbb{S}^8$ is homotopic to an entire rational map $\mathbb{S}^n \to \mathbb{S}^8$.*

References

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