Some series involving central binomial coefficients

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Using Maclaurin expansion $\arcsin = \sum\limits_{n=0}^{\infty} \frac{1}{2^{2n}}\binom{2n}{n}\frac{x^{2n+1}}{2n+1}$ and, for non-zero real variable x, formulas

$$\Re\Big(\arcsin\big(x\sqrt{i}\,\big)\Big) = \arctan\sqrt{\frac{\sqrt{1+x^4}-1}{x^2}},$$

$$\Im\Big(\arcsin\big(x\sqrt{i}\,\big)\Big) = \operatorname{arctanh}\sqrt{\frac{\sqrt{1+x^4}-1}{x^2}},$$

we obtain some series involving central binomial coefficients $\binom{2n}{n}$; see [1] for more details.

Theorem 1. *For $|x| \le 1$, we have*

$$\sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}}{4^n(2n+1)}\binom{2n}{n}x^{2n+1} = \sqrt{2}\arctan\Big(\frac{\sqrt{\sqrt{1+x^4}-1}}{x}\Big),$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}}{4^n}\binom{2n}{n}x^{2n} = \frac{\sqrt{2\sqrt{1+x^4}-2}}{\sqrt{1+x^4}\big(x^2-1+\sqrt{1+x^4}\big)},$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}n}{4^n}\binom{2n}{n}x^{2n} = \frac{x^2}{\sqrt{2}}\cdot\frac{3x^6-4x^4+5x^2-2+(3x^4-5x^2+2)\sqrt{1+x^4}}{(\sqrt{1+x^4}+x^2-1)^2\sqrt{(1+x^4)^3}\sqrt{\sqrt{1+x^4}-1}}.$$

Example 2. If $x=1$, $x=\sqrt{2}/2$, and $x=1/2$ then from Theorem 1 we have

$$\sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}}{4^n(2n+1)}\binom{2n}{n} = \sqrt{2}\operatorname{arccot}\sqrt{\delta}, \qquad \sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}}{4^n}\binom{2n}{n} = \frac{1}{\sqrt{2\delta}},$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}n}{4^n}\binom{2n}{n} = -\frac{\sqrt{\delta}}{4};$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}}{8^n(2n+1)}\binom{2n}{n} = 2\operatorname{arccot}\big(\alpha\sqrt{\alpha}\big), \qquad \sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}}{8^n}\binom{2n}{n} = \frac{2}{\sqrt{5\alpha}},$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}n}{8^n}\binom{2n}{n} = -\frac{\sqrt{5}}{25}\alpha^2\sqrt{\alpha};$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}}{16^n(2n+1)}\binom{2n}{n} = 2\sqrt{2}\operatorname{arccot}\sqrt{\sqrt{17}+4}, \qquad \sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}}{16^n}\binom{2n}{n} = \frac{2}{\sqrt{17}}\sqrt{\sqrt{17}-1},$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{\lfloor 3n/2\rfloor}n}{16^n}\binom{2n}{n} = -\frac{1}{17\sqrt{17}}\sqrt{17\sqrt{17}+47},$$

where $\alpha = (1+\sqrt{5})/2$ and $\delta = \sqrt{2}+1$ are the golden and silver ratios, respectively.

We will also establish connections with the Fibonacci and Lucas numbers. As usual, the Fibonacci numbers F_n and the Lucas numbers L_n are defined, for $n \in \mathbb{Z}$, through the recurrence $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, with initial values $F_0 = 0$, $F_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ with $L_0 = 2$, $L_1 = 1$. For negative subscripts, we have $F_{-n} = (-1)^{n-1} F_n$ and $L_{-n} = (-1)^n L_n$.

Theorem 3. *For any integer s,*

$$\sum_{n=0}^{\infty} \frac{(-1)^{\lfloor 3n/2 \rfloor}}{16^n(2n+1)} \binom{2n}{n} F_{2n+s} = -\frac{2\sqrt{10}}{5} \left(\alpha^{s-1} \arctan C_1 - \beta^{s-1} \arctan D_1\right),$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{\lfloor 3n/2 \rfloor}}{16^n(2n+1)} \binom{2n}{n} L_{2n+s} = -2\sqrt{2} \left(\alpha^{s-1} \arctan C_1 + \beta^{s-1} \arctan D_1\right);$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{\lfloor 3n/2 \rfloor}}{16^n} \binom{2n}{n} F_{2n+s} = \frac{8\sqrt{205}}{615} \left(\alpha^s C_2 + \beta^s D_2\right),$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{\lfloor 3n/2 \rfloor}}{16^n} \binom{2n}{n} L_{2n+s} = \frac{40\sqrt{41}}{615} \left(\alpha^s C_2 - \beta^s D_2\right);$$

$$\sum_{n=0}^{\infty} (-1)^{\lfloor 3n/2 \rfloor} \frac{n}{16^n} \binom{2n}{n} F_{2n+s} = \frac{\sqrt{205}}{226935} \left(\alpha^{s+2} C_3 - \beta^{s+2} D_3\right),$$

$$\sum_{n=0}^{\infty} (-1)^{\lfloor 3n/2 \rfloor} \frac{n}{16^n} \binom{2n}{n} L_{2n+s} = \frac{\sqrt{41}}{45387} \left(\alpha^{s+2} C_3 + \beta^{s+2} D_3\right),$$

where

$$C_1 = \beta \sqrt{\frac{\sqrt{78+6\sqrt{5}} - 8}{2}}, \quad D_1 = \alpha \sqrt{\frac{\sqrt{78-6\sqrt{5}} - 8}{2}},$$

$$C_2 = \frac{\sqrt{\sqrt{78+6\sqrt{5}} - 8}\,\sqrt{78-6\sqrt{5}}}{-5+\sqrt{5}+\sqrt{78+6\sqrt{5}}}, \quad D_2 = \frac{\sqrt{\sqrt{78-6\sqrt{5}} - 8}\,\sqrt{78+6\sqrt{5}}}{5+\sqrt{5}-\sqrt{78-6\sqrt{5}}},$$

$$C_3 = \frac{\left(148 - 112\sqrt{5} - \sqrt{78+6\sqrt{5}}\,(25 - 11\sqrt{5})\right)\sqrt{(78-6\sqrt{5})^3}}{\left(-5+\sqrt{5}+\sqrt{78+6\sqrt{5}}\right)^2 \sqrt{\sqrt{78+6\sqrt{5}} - 8}},$$

$$D_3 = \frac{\left(148 + 112\sqrt{5} - \sqrt{78-6\sqrt{5}}\,(25 + 11\sqrt{5})\right)\sqrt{(78+6\sqrt{5})^3}}{\left(5+\sqrt{5}-\sqrt{78+6\sqrt{5}}\right)^2 \sqrt{\sqrt{78-6\sqrt{5}} - 8}}.$$

Note that since $\binom{2n}{n} = (n+1)C_n$, where C_n are Catalan numbers, our results could be stated equivalently in terms of the Catalan numbers. Similar series were studied recently in [2, 3, 4, 5].

References

[1] K. Adegoke, R. Frontczak and T. Goy. Evaluation of some alternating series involving the binomial coefficients $C(4n, 2n)$. Preprint arXiv:2404.05770v1 [math.NT], 2024.

[2] K. Adegoke, R. Frontczak and T. Goy. Fibonacci–Catalan series. *Integers*, 22: #A110, 2022.

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