On non-topologizable semigroups

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In this paper we shall follow the semigroup terminology of [1, 2, 3, 4].

Throughout these abstract we always assume that all topological spaces involved are Hausdorff — unless explicitly stated otherwise.

Definition 1. Let X, Y and Z be topological spaces. A map $f\colon X\times Y\to Z$, $(x,y)\mapsto f(x,y)$, is called

(i) *right* [*left*] *continuous* if it is continuous in the right [left] variable; i.e., for every fixed $x_0\in X$ [$y_0\in Y$] the map $Y\to Z$, $y\mapsto f(x_0,y)$ [$X\to Z$, $x\mapsto f(x,y_0)$] is continuous;

(ii) *separately continuous* if it is both left and right continuous;

(iii) *jointly continuous* if it is continuous as a map between the product space $X\times Y$ and the space Z.

Definition 2. Let S be a non-void topological space which is provided with an associative multiplication (a semigroup operation) $\mu\colon S\times S\to S$, $(x,y)\mapsto \mu(x,y)=xy$. Then the pair (S,μ) is called

(i) a *right topological semigroup* if the map μ is right continuous, i.e., all interior left shifts $\lambda_s\colon S\to S$, $x\mapsto sx$, are continuous maps, $s\in S$;

(ii) a *left topological semigroup* if the map μ is left continuous, i.e., all interior right shifts $\rho_s\colon S\to S$, $x\mapsto xs$, are continuous maps, $s\in S$;

(iii) a *semitopological semigroup* if the map μ is separately continuous;

(iv) a *topological semigroup* if the map μ is jointly continuous.

We usually omit the reference to μ and write simply S instead of (S,μ). It goes without saying that every topological semigroup is also semitopological and every semitopological semigroup is both a right and left topological semigroup.

A topology τ on a semigroup S is called:

- a *semigroup* topology if (S,τ) is a topological semigroup;
- a *shift-continuous* topology if (S,τ) is a semitopological semigroup;
- an *left-continuous* topology if (S,τ) is a left topological semigroup;
- an *right-continuous* topology if (S,τ) is a right topological semigroup.

The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq=1$. The semigroup operation on $\mathscr{C}(p,q)$ is determined as follows:

$$q^kp^l\cdot q^mp^n=\begin{cases} q^{k-l+m}p^n, & \text{if } l<m;\\ q^kp^n, & \text{if } l=m;\\ q^kp^{l-m+n}, & \text{if } l>m.\end{cases}$$

We define the following subsets of the bicyclic monoid

$$\mathscr{C}_+(p,q)=\left\{q^ip^j\in\mathscr{C}(p,q)\colon i\leqslant j\right\} \quad \text{and} \quad \mathscr{C}_-(p,q)=\left\{q^ip^j\in\mathscr{C}(p,q)\colon i\geqslant j\right\}.$$

Proposition 3. *$\mathscr{C}_+(p,q)$ and $\mathscr{C}_-(p,q)$ are anti-isomorphic submonoids of $\mathscr{C}(p,q)$.*

Proposition 4. *Green's relations $\mathscr{R}$, $\mathscr{L}$, $\mathscr{J}$, $\mathscr{D}$ and $\mathscr{H}$ on monoids $\mathscr{C}_+(p,q)$ and $\mathscr{C}_-(p,q)$ coincide with the equality relation.*

Theorem 5. *Every Hausdorff left-continuous topology on the monoid $\mathscr{C}_+(p,q)$ is discrete.*

Theorem 6. *Every Hausdorff right-continuous topology on the monoid $\mathscr{C}_-(p,q)$ is discrete.*

Example 7. There exists a non-discrete locally compact semigroup T_1-topology τ on the monoid $\mathscr{C}_+(p,q)$.

Example 8. There exists a non-discrete compact shift-continuous T_1-topology τ on the monoid $\mathscr{C}_+(p,q)$.

Proposition 9. *If the monoid $\mathscr{C}_+(p,q)$ is a dense subsemigroup of a Hausdorff semitopological monoid S and $I = S \setminus \mathscr{C}_+(p,q) \neq \varnothing$ then I is a closed two-sided ideal of the semigroup S.*

Example 10. There exists a compact Hausdorff topological monoid S which contains the monoid $\mathscr{C}_+(p,q)$ as a dense submonoid.

Also, we discuss under which conditions a shift-continuous T_1-topology τ on the monoid $\mathscr{C}_+(p,q)$ is discrete.

References

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