Existence and non-existence of cohomogeneity one Einstein metrics

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A Riemannian metric g is Einstein if $\mathrm{Ric}(g) = \Lambda g$ for some constant Λ. A general existence theorem for homogeneous Einstein metrics was established in [WZ86]. It is natural to turn to the cohomogeneity one Einstein metrics, meaning that the principal orbit G/K is of codimension one. The cohomogeneity one condition reduces the Einstein equation to a system of ODEs. Previously known examples include [Pag78], [BB82], [KS86], [KS88], and [WW98]. Recently, we proved the existence of an Einstein metric on $\mathbb{HP}^{m+1}\sharp\overline{\mathbb{HP}}^{m+1}$ [Chi24], generalizing the result in [Böh98] to all higher dimensions.

We realize that the analytic techniques can be carried over to many other cohomogeneity one spaces. We develop two criteria to check the existence or non-existence of a cohomogeneity one Einstein metrics with a certain fixed principal orbit type. In particular, the principal orbit G/K is the total space of a sphere bundle over a singular orbit G/K, and both the fiber and the base space are irreducible. Each such a principal orbit is associated to a structural triple (d_1, d_2, A), where $d_1 = \dim(\mathsf{H}/\mathsf{K})$, $d_2 = \dim(\mathsf{G}/\mathsf{H})$ and $A > 0$ is a constant obtained from the O'neil tensor in the theory of Riemannian submersion. The corresponding cohomogeneity one space, denoted as M, is a double disk bundle, where G/K collapses to G/H on two ends. The Einstein metric is obtained from the ansatz

$$dt^2 + f_1^2(t)\, b|_{\mathfrak{h}/\mathfrak{k}} + f_2^2(t)\, b|_{\mathfrak{g}/\mathfrak{h}}\,, \tag{1}$$

where t parametrizes the 1-dimensional orbit space and b is a background metric.

Our existence theorem is the following.

Theorem 1. *For any (d_1, d_2) with $d_2 \geq d_1 \geq 2$, there exists a constant $\chi_{d_1,d_2} \in \left(0, \frac{d_2(d_2-1)^2}{d_1^2(d_1d_2-d_2+4)}\right]$ such that if G/K is a principal orbit with $A \in [0, \chi_{d_1,d_2})$, then there is at least one cohomogeneity one Einstein metrics on M.*

The constant χ_{d_1,d_2} is an algebraic function in (d_1, d_2), whose formula is very complicated in general. Nevertheless, we obtain many new examples of inhomogeneous Einstein metrics from previous works on homogeneous Einstein metrics including [DZ79], [WZ85], [Wan92], [DK08], [Nik16], [PZ21], and [LW24].

On the other hand, we also have the following non-existence theorem.

Theorem 2. *Define*

$$\Psi_{d_1,d_2} := \frac{(4(d_1-1)n^2 + d_2^2)(3n + d_1)}{(2n^2 + n + d_1)^2 d_1^2}\frac{d_2(d_2-1)^2}{4(d_1-1)}.$$

If G/K is a principal orbit with $(d_1, d_2) \notin \{(2,2),(2,3),(2,4)\}$ and $A \geq \Psi_{d_1,d_2}$, then there does not exists any G-invariant cohomogeneity one Einstein metrics on M from ansatz (1).

We find some examples of Theorem 2 from the classification in [DK08], including $\mathbb{OP}^2\sharp\overline{\mathbb{OP}}^2$ with $\mathrm{Spin}(9)/\mathrm{Spin}(7)$ as its principal orbit.

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