Stability of vertical minimal surfaces in three-dimensional sub-Riemannian manifolds

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A *sub-Riemannian manifold* is a smooth manifold M together with a completely non-integrable smooth distribution $\mathcal{H}$ on M (it is called a *horizontal distribution*) and a smooth field of Euclidean scalar products $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ on $\mathcal{H}$ (it is called a *sub-Riemannian metric*). In particular, $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ can be constructed as a restriction of some Riemannian metrics $\langle\cdot,\cdot\rangle$ on M to $\mathcal{H}$. Here we will assume that all sub-Riemannian structures are of this form. Let Σ be a smooth oriented surface in a three-dimensional sub-Riemannian manifold M. If N_h is the orthogonal projection of the unit normal field N of Σ (in the Riemannian sense) onto $\mathcal{H}$ and $d\Sigma$ is the Riemannian area form of Σ, then the *sub-Riemannian area* of a domain $D \subset \Sigma$ is defined as $A(D) = \int\limits_D |N_h|\, d\Sigma$. The *normal variation* of the surface Σ defined by a smooth function u is the map $\varphi\colon \Sigma \times I \to M\colon \varphi_s(p) = \exp_p(su(p)N(p))$, where I is an open neighborhood of 0 in $\mathbb{R}$ and $\exp_p$ is the Riemannian exponential map in p. Denote $A(s) = \int\limits_{\Sigma_s} |N_h|\, d\Sigma_s$, where $\Sigma_s = \varphi_s(\Sigma)$. Then $A'(0)$ is called the *first (normal) area variation* defined by φ, and $A''(0)$ is called the *second* one. A surface Σ is called *minimal* if $A'(0) = 0$ for any normal variations with compact support in $\Sigma \setminus \Sigma_0$, where $\Sigma_0 = \{p \in \Sigma \mid N_h(p) = 0\}$ is the *singular set* of Σ. A minimal surface Σ is called *stable* if $A''(0) \geq 0$ for any normal variations with compact support in $\Sigma \setminus \Sigma_0$. We will call a surface Σ in a three-dimensional sub-Riemannian manifold *vertical* if $T_p\Sigma \perp \mathcal{H}_p$ for each $p \in \Sigma$. In particular, for such surfaces $N_h = N$ and $\Sigma_0 = \varnothing$.

Proposition 1. *Let Σ be an oriented vertical surface in a three-dimensional sub-Riemannian manifold M. Then its first normal area variation defined by a smooth function u with compact support equals $A'(0) = -\int_\Sigma 2Hu\, d\Sigma$, where H is the Riemannian mean curvature of Σ. Thus, Σ is minimal in the sub-Riemannian sense if and only if it is minimal in the Riemannian sense.*

Proposition 2. *Let Σ be an oriented vertical minimal surface in a three-dimensional sub-Riemannian manifold M. Then its second normal area variation defined by a smooth function u with compact support equals*

$$A''(0) = \int_\Sigma -\left(X(u) - \langle\nabla_N X, N\rangle u\right)^2 + |\nabla_\Sigma u|^2 - \left(\mathrm{Ric}\,(N,N) + |B|^2\right) u^2\, d\Sigma,$$

where ∇ and Ric *are the Riemannian connection and the Ricci tensor of M respectively, X is the unit normal vector field of $\mathcal{H}$ (which is tangent to Σ because it is vertical), ∇_Σ and B are the Riemannian gradient and the second fundamental form of Σ respectively. It follows that if Σ is stable in the sub-Riemannian sense, it is also stable in the Riemannian sense.*

Let us discuss some examples of such surfaces. In all these examples M is a Lie group, $\mathcal{H}$ and $\langle\cdot,\cdot\rangle$ are left-invariant. In [2] it was shown that a complete connected minimal surface with the empty singular set (in particular, vertical) in the sub-Riemannian three-dimensional Heisenberg group is stable if and only if it is a vertical Euclidean plane. In [3] the authors considered the standard three-dimensional sphere with the horizontal distribution defined by the Hopf field X and showed that complete connected vertical minimal surfaces are Clifford tori. It is well-known that they are not stable in the Riemannian sense, hence also in the sub-Riemannian sense. In [1] we proved that in the solvable Lie group $\widetilde{E(2)}$, which is the universal covering of the proper motions group of the Euclidean plane, with the Euclidean metric and a left-invariant horizontal distribution all complete connected vertical minimal surfaces are Euclidean planes and standard helicoids. We showed that planes are stable in the sub-Riemannian sense, and it is known that helicoids are not stable in the Riemannian sense, hence also in the sub-Riemannian sense.

The three-dimensional Thurston geometry Sol is the space $\mathbb{R}^3$ with coordinates (x, y, z) and with the following orthonormal basis of left-invariant vector fields defined by its solvable Lie group structure:

$$X_1 = \frac{1}{\sqrt{2}}\left(e^{-z}\frac{\partial}{\partial x} + e^z\frac{\partial}{\partial y}\right),\ X_2 = \frac{1}{\sqrt{2}}\left(e^{-z}\frac{\partial}{\partial x} - e^z\frac{\partial}{\partial y}\right),\ X_3 = \frac{\partial}{\partial z}.$$

Note that $[X_2, X_3] = X_1$, so the left-invariant distribution $\mathcal{H}$ orthogonal to X_1 is completely non-integrable. Let us consider a sub-Riemannian structure on Sol such that $\mathcal{H}$ is horizontal. It then follows from the results of [4] that any complete connected vertical minimal surface in Sol after an isometry becomes either a Euclidean plane $z = C$ or a "helicoid"

$$(s, t) \mapsto \left(\frac{1}{\sqrt{2}}e^{-t}s + C_1, \frac{1}{\sqrt{2}}e^t s + C_2, t\right).$$

Using this description, we are able to prove the following.

Proposition 3. *All vertical minimal surfaces in Sol are stable in the sub-Riemannian sense and thus in the Riemannian sense.*

The three-dimensional Thurston geometry $\widetilde{SL(2,\mathbb{R})}$ can be described as the universal covering of the unit tangent bundle of the hyperbolic plane H^2 with the Sasaki metric, that is, the half-space $\{(x, y, z) \in \mathbb{R}^3 \mid y > 0\}$ with the following orthonormal basis of left-invariant vector fields with respect to its simple Lie group structure:

$$X_1 = y\left(-\sin z\frac{\partial}{\partial x} + \cos z\frac{\partial}{\partial y}\right) + \sin z\frac{\partial}{\partial z},\ X_2 = y\left(-\cos z\frac{\partial}{\partial x} - \sin z\frac{\partial}{\partial y}\right) + \cos z\frac{\partial}{\partial z},\ X_3 = \frac{\partial}{\partial z}.$$

In particular, $[X_1, X_2] = -X_3$, so the left-invariant distribution $\mathcal{H}$ orthogonal to X_3 is completely non-integrable. Consider a sub-Riemannian structure on this manifold such that $\mathcal{H}$ is horizontal. We than obtain the following description.

Theorem 4. *Any complete connected vertical minimal surface in $\widetilde{SL(2,\mathbb{R})}$ has either the parameterization $(s, t) \mapsto (C, s, t)$ or $(s, t) \mapsto \left(C_1 + \frac{1}{C_2}\sin C_2 s, -\frac{1}{C_2}\cos C_2 s, t\right)$ and so is a cylinder over a geodesic in H^2. All vertical minimal surfaces in $\widetilde{SL(2,\mathbb{R})}$ are stable in the sub-Riemannian sense and thus in the Riemannian sense.*

References

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