Rational factorization of Lax type flows in the space dual to the centrally extended Lie algebra of matrix super-integro-differential operators

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In papers of A. Prykarpatski, O. Hentosh and their coauthors the Lie-algebraic approach to the rational factorization of Lax type flows in spaces dual to certain operator Lie algebras and central extensions of some of them is developed for the central extension of the Lie algebra [1] of matrix super-integro-differential operators with one anticommuting variable.

Let us consider the Lie algebra g which consists of matrix super-integro-differential operators such as $\mathcal{A} := \mathbf{1}\partial^q + \sum_{p<2q} A_p D_\theta^p$, where $A_r \in C^\infty(\mathbb{S}\times\Lambda_1; gl(m|n))$, $gl(m|n))$ is a semi-simple Lie superalgebra of square supermatrices, $A_p = A_p(x,\theta) := A_p^0(x) + \theta A_p^1(x)$, $p \in \mathbb{Z}$, the supermatrices A_p are even for every even p and odd for every odd p, $\mathbf{1} \in gl(m+n)$ is an unit matrix, $q \in \mathbb{N}$, $\partial = \partial/\partial x$, $x \in \mathbb{S} \simeq \mathbb{R}/2\pi\mathbb{Z}$, $\theta \in \Lambda_1$, $\Lambda := \Lambda_0 \oplus \Lambda_1$ is a Grassmann algebra over the field $\mathbb{C} \subset \Lambda_0$, $D_\theta := \partial/\partial\theta + \theta\partial/\partial x$ is a superderivative, for which $D_\theta^2 = \partial/\partial x$, and $\partial/\partial\theta$ is a left partial derivative by the anticommuting variable θ, with the standard commutator $[.,.]$, acting by the rule:

$$[\mathcal{A},\, \mathcal{B}] = \mathcal{A}\circ\mathcal{B} - \mathcal{B}\circ\mathcal{A}, \quad \mathcal{A},\mathcal{B} \in \mathrm{g}, \tag{1}$$

where symbol "$\circ$" denotes the product of two matrix super-integro-differential operators (see [1]). The scalar product $(\mathcal{A},\, \mathcal{B}) = \int_{x\in\mathbb{S}} dx \int d\theta \; \mathrm{sSp}\,\mathrm{res}_{D_\theta}(\mathcal{A}\mathcal{B})$, where "$\mathrm{res}_{D_\theta}$" denotes the coefficient at D_θ^{-1} in the expansion of a matrix super-integro-differential operator and "sSp" is a supermatrix supertrace, being invariant with respect to the commutator (1), allows us to identify the dual space g^* to g, with the Lie algebra itself. The latter is splitting into the direct sum $\mathrm{g} := \mathrm{g}_+ \oplus \mathrm{g}_-$ of two its Lie subalgebras, where g_+ is a Lie subalgebra of formal polynomials by the superderivative operator with supermatrix-valued coefficients, and $\mathrm{g}_+^* \simeq \mathrm{g}_-$, $\mathrm{g}_-^* \simeq \mathrm{g}_+$.

One constructs the central extension $\hat{\mathrm{g}} := \tilde{\mathrm{g}} \oplus \mathbb{C}$ of parameterized Lie algebra $\tilde{\mathrm{g}} := \prod_{y\in\mathbb{S}} \mathrm{g}$ by the Maurer-Cartan two-cocycle on $\tilde{\mathrm{g}}$ such as $\omega_2(\mathcal{A},\mathcal{B}) = \int_{y\in\mathbb{S}} dy\, (\mathcal{A}, \partial\mathcal{B}/\partial y)$, where $\mathcal{A},\mathcal{B} \in \tilde{\mathrm{g}}$, with the commutator

$$[(\mathcal{A},d),\, (\mathcal{B},e)] = ([\mathcal{A},\mathcal{B}],\, \omega_2(\mathcal{A},\mathcal{B})), \quad (\mathcal{A},d),(\mathcal{B},e) \in \hat{\mathrm{g}},$$

and introduces another commutator on $\hat{\mathrm{g}}$ in the form

$$[(\mathcal{A},d),\, (\mathcal{B},e)]_{\mathcal{R}} = ([\mathcal{A},\mathcal{B}]_{\mathcal{R}},\, \omega_{2,\mathcal{R}}(\mathcal{A},\mathcal{B})), \tag{2}$$
$$[\mathcal{A},\mathcal{B}]_{\mathcal{R}} = [\mathcal{R}\mathcal{A},\mathcal{B}] + [\mathcal{A},\mathcal{R}\mathcal{B}], \quad \omega_{2,\mathcal{R}}(\mathcal{A},\mathcal{B}) = \omega_2(\mathcal{R}\mathcal{A},\mathcal{B}) + \omega_2(\mathcal{A},\mathcal{R}\mathcal{B}),$$

where $\mathcal{R} = (P_+ - P_-)/2$ and $P_\pm$ are projectors on the Lie subalgebras $\tilde{\mathrm{g}}_\pm$. On the space $\hat{\mathrm{g}}^*$, dual to $\hat{\mathrm{g}}$ with respect to the scalar product $((\mathcal{A},d),\,(\mathcal{B},e)) = \int_{y\in\mathbb{S}} dy\,(\mathcal{A},\mathcal{B}) + de$, the commutator (2) determines the Lie-Poisson bracket

$$\{\gamma,\mu\}_{\mathcal{R}}(l) = \int_{y\in\mathbb{S}} dy\,(l,\, [\nabla_l\gamma(l), \nabla_r\mu(l)]_{\mathcal{R}}) + c\omega_2(\nabla_l\gamma(l), \nabla_r\mu(l)) = (\nabla_l\gamma(l),\, \Theta\nabla_r\mu(l)), \tag{3}$$

where $\gamma,\mu \in \mathcal{D}(\tilde{\mathrm{g}}^*)$ are arbitrary smooth by Frechet functionals on $\tilde{\mathrm{g}}^* \simeq \tilde{\mathrm{g}}$, at a point $(l,c) \in \hat{\mathrm{g}}^*$. Here $l \in \tilde{\mathrm{g}}^*$ is some matrix super-integro-differential operator of order $q \in \mathbb{N}$, $c \in \mathbb{C}$, ∇_l, ∇_r are left and right gradient operators accordingly, $\Theta : T^*(\tilde{\mathrm{g}}^*) \to T(\tilde{\mathrm{g}}^*)$ is the Poisson operator generating the Lie-Poisson bracket (3) at a point $l \in \tilde{\mathrm{g}}^*$ and acting as $\Theta : \nabla\gamma(l) \mapsto -[l - c\mathbf{1}\partial/\partial y,\, (\nabla\gamma(l))_-] + [l - c\mathbf{1}\partial/\partial y,\, \nabla\gamma(l)]_-$ for any $\gamma \in \mathcal{D}(\tilde{\mathrm{g}}^*)$, the subscript "$-$" denotes the projection of the corresponding element from $\tilde{\mathrm{g}}$ on the Lie subalgebra $\tilde{\mathrm{g}}_- := \prod_{y\in\mathbb{S}} \mathrm{g}_-$, and $T(\tilde{\mathrm{g}}^*), T^*(\tilde{\mathrm{g}}^*)$ are tangent and cotangent spaces to $\tilde{\mathrm{g}}^*$.

The Casimir functionals $\gamma_j \in \mathcal{I}(\hat{\mathrm{g}}^*)$, $j \in \mathbb{N}$, of the central extension $\hat{\mathrm{g}}$, whose left gradients obey the equalities such that $[l - c\mathbf{1}\partial/\partial y,\, \nabla_l\gamma_j(l)] = 0$, where $\nabla_l\gamma_j(l) := \mathbf{1}\partial^j + \sum_{p<2j} A_{j,p} D_\theta^p$, $A_{j,p}$ are supermatrix-valued functions of suitable parity, $j \in \mathbb{N}$, $p \in \mathbb{Z}$, $p < 2j$, at a point $(l, c) \in \hat{\mathrm{g}}^*$, and the $\mathcal{R}$-deformed Lie-Poisson bracket (3) give us the hierarchy of Lax type Hamiltonian flows on $\tilde{\mathrm{g}}^* \simeq \tilde{\mathrm{g}}$:

$$dl/dt_j = [(\nabla_l \gamma_j(l)))_+, \, l - c\mathbf{1}\partial/\partial y], \quad j \in \mathbb{N}, \quad t_j \in \mathbb{R},$$

where the subscript "+" denotes the projection of the corresponding element from $\tilde{\mathrm{g}}$ on the Lie subalgebra $\tilde{\mathrm{g}}_+ := \prod_{y \in \mathbb{S}} \mathrm{g}_+$. One considers another hierarchy of Lax type Hamiltonian flows on the dual space $\tilde{\mathrm{g}}^*$:

$$d\tilde{l}/dt_j = [(\nabla_l \gamma_j(\tilde{l}))_+, \, \tilde{l} - c\mathbf{1}\partial/\partial y], \quad j \in \mathbb{N}, \quad t_j \in \mathbb{R},$$

for some matrix super-integro-differential operator $\tilde{l} \in \tilde{\mathrm{g}}^*$ of order $q \in \mathbb{N}$, which is related with the operator $l \in \tilde{\mathrm{g}}^*$ by the generalized gauge transformation

$$\tilde{l}(0) - c\mathbf{1}\partial/\partial y = \mathcal{B}(0)^{-1}(l(0) - c\mathbf{1}\partial/\partial y)\mathcal{B}(0), \tag{4}$$

where $\mathcal{B}(0) \in \tilde{\mathrm{g}}_+$ is a matrix superdifferential operator of order $s \in \mathbb{N}$ with constant coefficients, at the initial moment of the time $t_j \in \mathbb{R}$ for every $j \in \mathbb{N}$.

Theorem 1. *If for every $j \in \mathbb{N}$ at the initial moment of the time $t_j \in \mathbb{R}$ matrix super-integro-differential operators $l, \tilde{l} \in \tilde{\mathrm{g}}^*$ of order $q \in \mathbb{N}$ are related by the relationship* (4)*, there exist such matrix superdifferential operators of orders $q + s$ and s accordingly, where $s \in \mathbb{Z}_+$, $s < q$, that the equalities*

$$l = \mathcal{A}\mathcal{B}^{-1}, \quad \tilde{l} = \mathcal{B}^{-1}(\mathcal{A} - c\partial\mathcal{B}/\partial y) \tag{5}$$

hold. The operators $\mathcal{A}, \mathcal{B} \in \tilde{\mathrm{g}}_+$ satisfy the following systems of evolution equations

$$\begin{aligned} d\mathcal{A}/dt_j &= (\nabla_l \gamma_j(l))_+\mathcal{A} - \mathcal{A}(\nabla_l \gamma_j(\tilde{l}))_+ - c(\partial(\nabla_l \gamma_j(l))_+/\partial y)\mathcal{B}, \\ d\mathcal{B}/dt_j &= (\nabla_l \gamma_j(l))_+\mathcal{B} - \mathcal{B}(\nabla_l \gamma_j(\tilde{l}))_+, \quad j \in \mathbb{N}, \end{aligned} \tag{6}$$

which possess an infinite sequence of the conservation laws $H_j \in \mathcal{D}(\tilde{\mathrm{g}}_+ \times \tilde{\mathrm{g}}_+)$, $j \in \mathbb{N}$, in the form

$$H_j(\mathcal{A}, \mathcal{B}) := \gamma_j(l)|_{l = \mathcal{A}\mathcal{B}^{-1}} = \gamma_j(\tilde{l})|_{\tilde{l} = \mathcal{B}^{-1}(\mathcal{A} - c\partial\mathcal{B}/\partial y)}.$$

The equalities (5) determine the Backlund transformation

$$P : (\mathcal{A}, \mathcal{B}) \in \tilde{\mathrm{g}} \times \tilde{\mathrm{g}} \mapsto (l, \tilde{l}) \in \tilde{\mathrm{g}}^* \oplus \tilde{\mathrm{g}}^*. \tag{7}$$

Theorem 2. *For every $j \in \mathbb{N}$ the system of evolution equations* (6)*, given on the subspace $\tilde{\mathrm{g}}_+ \times \tilde{\mathrm{g}}_+ \subset \tilde{\mathrm{g}} \times \tilde{\mathrm{g}}$, is Hamiltonian with respect to the Poisson bracket $\{.,.\}_{\mathcal{L}}$, which arises as a reduction of the Poisson bracket $\{.,.\}_{\bar{\mathcal{L}}}$ with the corresponding Poisson operator $\bar{\mathcal{L}} : T^*(\tilde{\mathrm{g}} \times \tilde{\mathrm{g}}) \to T(\tilde{\mathrm{g}} \times \tilde{\mathrm{g}})$ such as $\bar{\mathcal{L}} = (P')^{-1}(\Theta \oplus \tilde{\Theta})(P'^*)^{-1}$, where $\Theta, \tilde{\Theta} : T^*(\tilde{\mathrm{g}} \times \tilde{\mathrm{g}}) \to T(\tilde{\mathrm{g}} \times \tilde{\mathrm{g}})$ are Poisson operators generating the Lie-Poisson bracket $\{.,.\}_{\mathcal{R}}$ at points $l, \tilde{l} \in \tilde{\mathrm{g}}^*$ accordingly, $P'^* : T^*(\tilde{\mathrm{g}}^* \oplus \tilde{\mathrm{g}}^*) \to T^*(\tilde{\mathrm{g}} \times \tilde{\mathrm{g}})$ is an operator adjoint to the Frechet derivative $P' : T(\tilde{\mathrm{g}} \times \tilde{\mathrm{g}}) \to T(\tilde{\mathrm{g}}^* \oplus \tilde{\mathrm{g}}^*)$ of the Backlund transformation* (7) *and $(P'^*)^{-1}$ is an operator inverse to P', on $\tilde{\mathrm{g}}_+ \times \tilde{\mathrm{g}}_+$ and the Hamiltonians $\bar{H}_j \in \mathcal{D}(\tilde{\mathrm{g}}_+ \times \tilde{\mathrm{g}}_+)$, $j \in \mathbb{N}$:*

$$\bar{H}_j(\mathcal{A}, \mathcal{B}) := \gamma_j(l)|_{l = \mathcal{A}\mathcal{B}^{-1}} + \gamma_j(\tilde{l})|_{\tilde{l} = \mathcal{B}^{-1}(\mathcal{A} - c\partial\mathcal{B}/\partial y)}.$$

The reductions of the hierarchy (6) on the coadjoint action orbits for the central extension $\hat{\mathrm{g}}$ with taking into account the Backlund transformation (7) lead to new integrable hierarchies of nonlinear dynamical systems on matrix functional supermanifolds of two commuting and one anticommuting independent variables, being Hamiltonian ones and possessing infinite sequences of conservation laws.

References

[1] Oksana E. Hentosh. Lie-algebraic structure of the Lax-integrable (2|1+1)-dimensional supersymmetric matrix dynamical systems. *Ukr. Math. J.*, 69(10) : 1537–1560, 2018.