On boundary controllability problems for the heat equation with variable coefficients on a half-axis

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Consider the control system for the heat equation on a half-axis:

$$w_t = \frac{1}{\rho}(kw_x)_x + \gamma w, \qquad x \in \mathbb{R}_+,\ t \in (0,T), \tag{1}$$

$$\left(\sqrt{k/\rho}\,w_x\right)\Big|_{x=0} = u, \qquad t \in (0,T), \tag{2}$$

$$w(\cdot,0) = w^0, \qquad x \in \mathbb{R}_+, \tag{3}$$

where $\mathbb{R}_+ = (0,+\infty)$; T is a positive constant; ρ, k, γ, w^0 are given functions; $u \in L^\infty(0,T)$ is a control. We assume ρ and k are positive on $[0,+\infty)$, $\rho, k \in C^1[0,+\infty)$, $(\rho k) \in C^2[0,+\infty)$, $(\rho k)'(0) = 0$, and

$$\sigma(x) = \int_0^x \sqrt{\rho(\mu)/k(\mu)}\,d\mu \to +\infty \qquad \text{as } x \to +\infty.$$

Moreover, we assume

$$(Q_2(\rho,k) - \gamma) \in L^\infty(0,+\infty) \bigcap C^1[0,+\infty) \qquad \text{and} \qquad \sigma\sqrt{\rho/k}\,(Q_2(\rho,k)-\gamma) \in L^1(0,+\infty),$$

where $Q_2(\rho,k) = \sqrt{k/\rho}\big(Q_1(\rho,k)\big)' + \big(Q_1(\rho,k)\big)^2$, $Q_1(\rho,k) = \sqrt{k/\rho}(k\rho)'/(4k\rho)$.

Control system (1)–(3) is considered in modified Sobolev spaces. Let $\varphi \in L^2_{\text{loc}}(\mathbb{R}_+)$. We define the modified derivative $\mathbb{D}_{\rho k}$ by the rule

$$\mathbb{D}_{\rho k}\varphi = \sqrt{k/\rho}\,\varphi' + Q_1(\rho,k)\varphi.$$

If, in addition, $\mathbb{D}_{\rho k}\varphi \in L^2_{\text{loc}}(\mathbb{R}_+)$ and $(\mathbb{D}_{\rho k}\varphi)' \in L^2_{\text{loc}}(\mathbb{R}_+)$ we can consider $\mathbb{D}^2_{\rho k}\varphi$:

$$\mathbb{D}^2_{\rho k}\varphi = \frac{1}{\rho}\big(k\varphi'\big)' + Q_2(\rho,k)\varphi.$$

Obviously, $\mathbb{D}^m_{\rho k}\varphi = \varphi^{(m)}$ if $\rho = k = 1$, $m = 0,1$.

Denote

$$L^2_\rho(\mathbb{R}_+) = \{f \in L^2_{\text{loc}}(\mathbb{R}_+) \mid \sqrt{\rho}f \in L^2(\mathbb{R}_+)\}$$

with the norm $\|f\|_{L^2_\rho(\mathbb{R}_+)} = \|\sqrt{\rho}f\|_{L^2(\mathbb{R}_+)}$, $f \in L^2_\rho(\mathbb{R}_+)$.

Now consider the modified Sobolev spaces

$$\overset{\circ}{\mathbb{H}}{}^0 = L^2_\rho(\mathbb{R}_+), \quad \overset{\circ}{\mathbb{H}}{}^1 = \{\varphi \in L^2_\rho(\mathbb{R}_+) \mid \mathbb{D}_{\rho k}\varphi \in L^2_\rho(\mathbb{R}_+) \text{ and } \varphi(0^+) = 0\}$$

with the norm

$$[\![\varphi]\!]^p = \left(\sum_{m=0}^p \binom{p}{m}\left(\|\mathbb{D}^m_{\rho k}\varphi\|_{L^2_\rho(\mathbb{R}_+)}\right)^2\right)^{1/2}, \quad \varphi \in \overset{\circ}{\mathbb{H}}{}^p, \quad p = 0,1,$$

and the dual space $\overset{\circ}{\mathbb{H}}{}^{-p} = \left(\overset{\circ}{\mathbb{H}}{}^{p}\right)^*$, $p = 0, 1$, with the norm associated with the strong topology of the adjoint space.

In control system (1)–(3), we suppose $\left(\frac{d}{dt}\right)^p w : [0, T] \to \overset{\circ}{\mathbb{H}}{}^{1-2p}$, $p = 0, 1$; $w^0 \in \overset{\circ}{\mathbb{H}}{}^1$.

Let $T > 0$, $w^0 \in \overset{\circ}{\mathbb{H}}{}^1$. By $\mathcal{R}_T(w^0)$, denote the set of all states $w^T \in \overset{\circ}{\mathbb{H}}{}^1$ for which there exists a control $u \in L^\infty(0, T)$ such that there exists a unique solution w to (1)–(3) and $w(\cdot, T) = w^T$.

Definition 1. A state $w^0 \in \overset{\circ}{\mathbb{H}}{}^1$ is said to be *null-controllable* with respect to system (1)–(3) in a given time $T > 0$ if $0 \in \mathcal{R}_T(w^0)$.

Definition 2. A state $w^0 \in \overset{\circ}{\mathbb{H}}{}^1$ is said to be *approximately controllable* to a state $w^T \in \overset{\circ}{\mathbb{H}}{}^1$ with respect to system (1)–(3) in a given time $T > 0$ if $w^T \in \overline{\mathcal{R}_T(w^0)}$, where the closure is considered in the space $\overset{\circ}{\mathbb{H}}{}^1$.

Consider also the control system with the simplest heat operator (the case $\rho = k = 1$, $\gamma = 0$):

$$z_t = z_{yy}, \qquad y \in \mathbb{R}_+,\ t \in (0, T), \tag{4}$$

$$z_y(0, \cdot) = v, \qquad t \in (0, T), \tag{5}$$

$$z(\cdot, 0) = z^0, \qquad y \in \mathbb{R}_+, \tag{6}$$

where $v \in L^\infty(0, T)$ is a control, $\left(\frac{d}{dt}\right)^m z : [0, T] \to H^{1-2m}$, $m = 0, 1$, $z^0 \in H^1$. Here H^p, $p = -1, 0, 1$, are the Sobolev spaces.

Controllability problems for system (4)–(6) were investigated in [1].

To study controllability problems for system (1)–(3), we use the transformation operator $\widehat{\mathbb{T}} : H^{-1} \to \overset{\circ}{\mathbb{H}}{}^{-1}$. It was introduced and studied in [2]. In particular, it has been proved therein that $\widehat{\mathbb{T}}$ is a continuous one-to-one mapping between the spaces H^p and $\overset{\circ}{\mathbb{H}}{}^p$, $p = -1, 0, 1$.

In the present talk, we prove that the transformation operator $\widehat{\mathbb{T}}$ is one-to-one mapping between the sets of the solutions to system (4)–(6) and to system (1)–(3). The application of the operator $\widehat{\mathbb{T}}$ allows us to conclude that the control system (1)–(3) replicates the controllability properties of the control system (4)–(6) and vice versa. A relation between controls u and v is also found. Thus, using obtained results for control system (4)–(6), we obtain the following main results for control system (1)–(3).

Theorem 3. *If a state $w^0 \in \overset{\circ}{\mathbb{H}}{}^1$ is null-controllable with respect to system (1)–(3) in a time $T > 0$, then $w^0 = 0$.*

Theorem 4. *Each state $w^0 \in \overset{\circ}{\mathbb{H}}{}^1$ is approximately controllable to any target state $w^T \in \overset{\circ}{\mathbb{H}}{}^1$ with respect to system (1)–(3) in a given time $T > 0$.*

All obtained results have been published in [3].

References

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