# On explicit reconstruction of real algebraic maps locally like moment maps

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Real algebraic geometry studies real algebraic varieties. Nash and Tognoli's theory shows that a smooth closed manifold is realized as the zero set of a real algebraic map and non-singular. It is also well-known that smooth maps between non-singular real algebraic manifolds are approximated by real algebraic maps in general. For related history and terminologies and notions, see [5] for example.

Our study is on explicit construction of real algebraic sets and maps. The $k$-dimensional unit sphere $S^k$ in the $(k+1)$-dimensional real affine space $\mathbb{R}^{k+1}$ and its canonical embedding into higher dimensional real affine spaces (and the compositions with canonical projections) give simplest examples. In general, it is very difficult to give real algebraic sets and important real algebraic maps on them very explicitly. Here, we reconstruct real algebraic maps generalizing the canonical projections of the unit spheres $S^k \subset \mathbb{R}^{k+1}$ from given regions in the target spaces. We discuss this from the viewpoint of global singularity theory and differential topology of manifolds.

**Theorem 1.** [2] *Let $l_1, l_2$ and $n$ be positive integers. Let $D \subset \mathbb{R}^n$ be an open subset. Let $\{S_j\}_{j=1}^{l_1}$ be a family of non-singular real algebraic hypersurfaces of $\mathbb{R}^n$. Let $S_j$ be also the zero set of a real polynomial $f_j$. We also assume the following.*

1. *It holds that $D \bigcap \bigcup_{j=1}^{l_1} S_j = \emptyset$, that $\overline{D} \bigcap S_j \neq \emptyset$ for $1 \leq j \leq l_1$ and that $\overline{D} - D \subset \bigcup_{j=1}^{l_1} S_j$. For any small open neighborhood $U_D$ of $\overline{D}$, $D = U_D \bigcap \bigcap_{j=1}^{l_1} \{x \mid f_j(x) > 0\}$ and $\overline{D} = U_D \bigcap \bigcap_{j=1}^{l_1} \{x \mid f_j(x) \geq 0\}$.*
2. *Let $\{i_j\}_{j=1}^{i_0}$ be an increasing sequence of integers such that $1 \leq i_j \leq l_1$. Let $p \in \bigcap_{j=1}^{i_0} S_{i_j} \bigcap \overline{D}$ and for any increasing sequence $\{{i_j}'\}_{j=1}^{i_0+1}$ of integers satisfying $1 \leq {i_j}' \leq l_1$ and containing $\{i_j\}_{j=1}^{i_0}$ as a subsequence, $p \notin \bigcap_{j=1}^{i_0+1} S_{{i_j}'}$ hold. Let $e_j : S_j \rightarrow \mathbb{R}^n$ denote the canonical embedding. Assume that $\bigcap_{j=1}^{i_0} S_{i_j}$ is an $(n-i_0)$-dimensional smooth submanifold of $\mathbb{R}^n$ with no boundary and let $e_{\{i_j\}_{j=1}^{i_0}} : \bigcap_{j=1}^{i_0} S_{i_j} \rightarrow \mathbb{R}^n$ denote the canonical embedding. Then the intersection $\bigcap_{j=1}^{i_0} de_j(T_p S_j)$ and the image $de_{\{i_j\}_{j=1}^{i_0}}(T_p(\bigcap_{j=1}^{i_0} S_{i_j}))$ of the differential of $e_{\{i_j\}_{j=1}^{i_0}}$ at $p$ always agree.*
3. *There exists a map $m_{l_1,l_2}$ which maps each integer $1 \leq i \leq l_1$ to an integer $1 \leq i' \leq l_2$, which is a surjection to the set of all integers $1 \leq j \leq l_2$, and whose restriction to an increasing sequence $\{i_j\}_{j=1}^{i_0}$ making $\bigcap_{j=1}^{i_0} S_{i_j} \bigcap \overline{D}$ a non-empty set is always injective.*

*Let $m_{l_2}$ be a map mapping each integer $1 \leq i \leq l_2$ to a non-negative integer. Let $m := n + \Sigma_{j=1}^{l_2} m_{l_2}(j)$. Then there exist an $m$-dimensional non-singular real algebraic manifold $M \subset \mathbb{R}^{m+l_2}$ and a real algebraic map $f : M \rightarrow \mathbb{R}^n$ such that $f(M) = \overline{D}$, that the image of the singular set is $\overline{D} - D$, and that each preimage $f^{-1}(p)$ ($p \in \overline{D}$) is a single-point set or a product of spheres and at most $(m-n)$-dimensional.*

We present the case where hypersurfaces $S_j$ do not intersect and $l_2 = 1$. Here $M := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^{m-n+1} = \mathbb{R}^{m+1} \mid \prod_{j=1}^{l_1}(f_j(x)) - {||y||}^2 = 0\}$. The map $f : M \rightarrow \mathbb{R}^n$ is the composition of the canonical embedding into $\mathbb{R}^{m+1}$ with the canonical projection. In the case $\overline{D} := D^n$, the $n$-dimensional unit sphere, we have the canonical projection of the unit sphere $S^m \subset \mathbb{R}^{m+1}$ into $\mathbb{R}^n$. This is already done in [1, 3]. In the case the hypersurfaces $S_j$ do not intersect, we have so-called *special generic* maps presented in [4]. See also [6] for special generic maps in the theory of differential topology of manifolds. Our result reconstructs real algebraic maps locally like moment maps.

As an application, Theorem 2 presents explicit families of real algebraic functions which are compositions of maps into $\mathbb{R}^2$ obtained through Theorem 1 with the canonical projection ([2]). This is seen as extending some result of [1]: we have studied cases where the hypersurfaces $S_j$ do not intersect. We define the *Reeb graph* $W_c$ of a smooth function $c$ on a closed manifold $X$. Let $\sim_c$ be the following equivalence relation on $X$: $x_1 \sim_c x_2$ if and only if $x_1$ and $x_2$ are in a same connected component of a preimage $c^{-1}(y)$ ($y \in \mathbb{R}$). Let $q_c : X \to W_c := X/\sim_c$ denote the quotient map. The vertex set is the set of all points $v$ such that ${q_c}^{-1}(v)$ contain some critical points of $c$. See also [7] for example.

**Theorem 2.** [2] *In Theorem 1, let $n = 2$, $l_1 := l_{0,1}$, $\{S_j := S_{0,j}\}_{j=1}^{l_1}$ consist of mutually disjoint circles and $D := D_0 \subset \mathbb{R}^2$ be a connected and bounded open set surrounded by the disjoint union $\sqcup_{j=1}^{l_{0,1}} S_j$.*

*We construct a map $f := f_{M_0} : M := M_0 \to \mathbb{R}^2$ by Theorem 1. We consider the function $f_{0,M_0} := \pi_{2,1} \circ f_{M_0}$ where $\pi_{2,1} : \mathbb{R}^2 \to \mathbb{R}$ is the canonical projection. Let $m \geq 4$ be an integer. For each non-negative integer $k$, we can consider some situation enjoying the following and apply Theorem 1.*

*(1) $n = 2$. $l_1 := l_{0,1} + k$. $l_2 := 2$.*

*(2) The family $\{S_j\}_{j=1}^{l_1}$ is a family of circles enjoying the following.*

*(a) For $1 \leq j \leq l_{0,1}$, $S_j := S_{0,j}$ and the family $\{S_{l_{0,1}+j}\}_{j=1}^{k}$ consists of mutually disjoint and sufficiently small circles in $\mathbb{R}^2$ centered at points in some circles in $\{S_j := S_{0,j}\}_{j=1}^{l_1}$.*

*(b) For each $S_{l_{0,1}+j_2}$ $(1 \leq j_2 \leq k)$, there exists a unique circle $S_{0,j_1}$ such that $S_{l_{0,1}+j_2} \bigcap S_{0,j_1} \neq \emptyset$. Furthermore, the non-empty intersection is a two-point set.*

*(3) The open set $D \subset \mathbb{R}^2$ is the connected and bounded component of the complementary set of $\bigcup_{j=1}^{l_1} S_j$ in $\mathbb{R}^2$ which is also a subset of $D_0$.*

*Furthermore, choosing suitable situations, we have a family $\{f := f_{M_k} : M := M_k \to \mathbb{R}^2\}_{k=0}^{\infty}$ of real algebraic maps on $m$-dimensional closed and connected manifolds enjoying the following properties.*

*(4) The Reeb graph $W_{f_{0,M_k}}$ of the function $f_{0,M_k} := \pi_{2,1} \circ f_{M_k}$ collapses to the Reeb graph $W_{f_{0,M_0}}$.*

*(5) The graphs $W_{f_{0,M_{k_1}}}$ and $W_{f_{0,M_{k_2}}}$ are not isomorphic for distinct integers $k_1$ and $k_2$.*

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