Group classification of Kolmogorov backward equations with power diffusivity

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We carry out the complete group classification of the class $\mathcal{F}$ of Kolmogorov backward equations with power diffusivity

$$\mathcal{F}_{\alpha\beta}\colon\ u_t + xu_y = |x-\alpha|^\beta u_{xx},$$

where α and β are arbitrary real parameters, by solving the same problem for the class $\mathcal{F}'$ of the equations of the form

$$\mathcal{F}'_\beta\colon\quad u_t + xu_y = |x|^\beta u_{xx},$$

with β remains to be the only arbitrary element in the class $\mathcal{F}'$. Using the modified version of the direct method, we compute the equivalence groupoids $\mathcal{G}^\sim_{\mathcal{F}}$ and $\mathcal{G}^\sim_{\mathcal{F}'}$ of the classes $\mathcal{F}$ and $\mathcal{F}'$, respectively, and consequently show that the class $\mathcal{F}'$ is semi-normalized in the usual sense. The modification of the direct method is based on embedding both the classes $\mathcal{F}$ and $\mathcal{F}'$ into the class $\bar{\mathcal{F}}$ of ultraparabolic (1+2)-dimensional Fokker–Planck equations of the form

$$u_t + B(t,x,y)u_y = A^2(t,x,y)u_{xx} + A^1(t,x,y)u_x + A^0(t,x,y)u + C(t,x,y),$$

where the tuple $\bar\theta := (B, A^2, A^1, A^0, C)$ of arbitrary elements of the class $\bar{\mathcal{F}}$ runs through the solution set of the system of the inequalities $A^2\neq 0$ and $B_x\neq 0$ with no restrictions on A^0, A^1 and C. The equivalence groupoid of of the class $\bar{\mathcal{F}}$ was described in [1, 2] via presenting the equivalence group of this class and stating that it is normalized, see [3] for required notions, results and further references. We use the known determining equations for admissible transformations within the superclass $\bar{\mathcal{F}}$ as the known principal constraints for admissible transformations within the classes $\mathcal{F}$ and $\mathcal{F}'$. After explicitly constructing the groupoids $\mathcal{G}^\sim_{\mathcal{F}}$ and $\mathcal{G}^\sim_{\mathcal{F}'}$, it is easy to show that the group classification of the class $\mathcal{F}$ reduces to that of the class $\mathcal{F}'$.

The class $\mathcal{F}'$ admits a distinguished discrete equivalence transformation

$$\mathscr{J}\colon\quad \tilde t = y\,\mathrm{sgn}\,x,\quad \tilde x = \frac{1}{x},\quad \tilde y = t\,\mathrm{sgn}\,x,\quad \tilde u = \frac{u}{x},\quad \tilde\beta = 5-\beta,$$

which turns out to be the only point equivalence transformation essential for carrying out the group classification of this class modulo the $\mathcal{G}^\sim_{\mathcal{F}'}$-equivalence.

The following chain of assertions provides the complete solutions to the group classification problems for the classes $\mathcal{F}$ and $\mathcal{F}'$.

Theorem 1. (i) *The point transformations $\mathscr{S}(c_1)$: $(\tilde t,\tilde x,\tilde y,\tilde u,\tilde\beta,\tilde\alpha) = (t, x+c_1, y+c_1t, u, \beta, \alpha+c_1)$ where c_1 is arbitrary constant, constitute a one-parameter group of equivalence transformations of the class $\mathcal{F}$.*

(ii) *The wide family of admissible transformations $\mathcal{S}_{\alpha\beta} := \big((\alpha,\beta), \pi_*\mathscr{S}(-\alpha), (0,\beta)\big)$ of the class $\mathcal{F}$ from the action groupoid of its equivalence group maps this class onto the class $\mathcal{F}'$ interpreted as a subclass of $\mathcal{F}$.*

(iii) *The point transformation $\mathscr{J}$ is a (discrete) equivalence transformation of the class $\mathcal{F}'$.*

(iv) *The class $\mathcal{F}'$ is semi-normalized with respect to the discrete equivalence subgroup generated by $\mathscr{J}$. In other words, the equivalence groupoid $\mathcal{G}^{\sim}_{\mathcal{F}'}$ of $\mathcal{F}'$ is the Frobenius product of the action groupoid of this subgroup and the fundamental equivalence groupoid $\mathcal{G}^{\rm f}_{\mathcal{F}'}$ of $\mathcal{F}'$.*

Corollary 2. (i) *Different equations $\mathcal{F}'_\beta$ and $\mathcal{F}'_{\tilde\beta}$ are similar with respect to point transformations if and only if $\beta+\tilde\beta=5$.*

(ii) *Equations $\mathcal{F}_{\alpha\beta}$ and $\mathcal{F}_{\tilde\alpha\tilde\beta}$ are similar with respect to point transformations if and only if either $\tilde\beta=\beta$ or $\beta+\tilde\beta=5$.*

(iii) *The equivalence groupoid $\mathcal{G}^{\sim}_{\mathcal{F}}$ of $\mathcal{F}$ is generated by admissible transformations $\mathcal{S}_{\alpha\beta}$ and elements of $\mathcal{G}^{\sim}_{\mathcal{F}'}$. More specifically, for each admissible transformation $\big((\alpha,\beta),\Phi,(\tilde\alpha,\tilde\beta)\big)$ of $\mathcal{F}$, we have $\Phi=\pi_*\mathscr{S}(\tilde\alpha)\circ\breve\Phi\circ\pi_*\mathscr{S}(-\alpha)$ for some point transformation $\breve\Phi$ with $(\beta,\breve\Phi,\tilde\beta)\in\mathcal{G}^{\sim}_{\mathcal{F}'}$.*

Theorem 3. *The kernel Lie invariance algebra $\mathfrak{g}^{\cap}_{\mathcal{F}'}$ of the equations from the class $\mathcal{F}'$ is*

$$\mathfrak{g}^{\cap}_{\mathcal{F}'}=\langle\mathcal{P}^t,\mathcal{P}^y,\mathcal{I},(tx-y)\partial_u,x\partial_u,\partial_u\rangle,\quad\text{where}\quad \mathcal{P}^t:=\partial_t,\quad \mathcal{P}^y:=\partial_y,\quad \mathcal{I}:=u\partial_u.$$

Any equation $\mathcal{F}'_\beta$ from $\mathcal{F}'$ is invariant with respect to the algebra

$$\mathfrak{g}^{\rm gen}_\beta=\langle\mathcal{P}^t,\mathcal{P}^y,\mathcal{I},\mathcal{D}^\beta,\mathcal{Z}(f^\beta)\rangle\quad\text{with}\quad \mathcal{D}^\beta:=(2-\beta)t\partial_t+x\partial_x+(3-\beta)y\partial_y,\ \mathcal{Z}(f^\beta):=f^\beta\partial_u,$$

where the parameter function $f^\beta=f^\beta(t,x,y)$ runs through the solution set of this equation, and $\beta\in(-\infty,5/2]$ modulo the $G^{\sim}_{\mathcal{F}'}$-equivalence. the maximal Lie invariance algebra $\mathfrak{g}_\beta$ of the equation $\mathcal{F}_\beta$ coincides with $\mathfrak{g}^{\rm gen}_\beta$ if and only if $\beta\in\mathbb{R}\setminus\{0,2,3,5\}$. A complete list of $G^{\sim}_{\mathcal{F}'}$-inequivalent essential Lie symmetry extensions in the class $\mathcal{F}'$ is exhausted by the following cases:

$$\beta=2\colon\quad \mathfrak{g}_2=\mathfrak{g}^{\rm gen}_2\dotplus\langle\mathcal{K}_2\rangle\quad\text{with}\quad \mathcal{K}_2=2xy\partial_x+y^2\partial_y-xu\partial_u,$$

$$\begin{aligned}\beta=0\colon\quad &\mathfrak{g}_0=\mathfrak{g}^{\rm gen}_0\dotplus\langle\mathcal{K}_0,\mathcal{P}^3,\mathcal{P}^2,\mathcal{P}^1\rangle\quad\text{with}\\ &\mathcal{K}_0=t^2\partial_t+(tx+3y)\partial_x+3ty\partial_y-(x^2+2t)u\partial_u,\\ &\mathcal{P}^3=3t^2\partial_x+t^3\partial_y+3(y-tx)u\partial_u,\quad \mathcal{P}^2=2t\partial_x+t^2\partial_y-xu\partial_u,\quad \mathcal{P}^1=\partial_x+t\partial_y.\end{aligned}$$

Corollary 4. *The kernel Lie invariance algebra $\mathfrak{g}^{\cap}_{\mathcal{F}}$ of the equations from the class $\mathcal{F}$ coincides with that for the class $\mathcal{F}'$, $\mathfrak{g}^{\cap}_{\mathcal{F}}=\mathfrak{g}^{\cap}_{\mathcal{F}'}$ Any equation $\mathcal{F}_{\alpha\beta}$ from $\mathcal{F}$ is invariant with respect to the algebra*

$$\mathfrak{g}^{\rm gen}_{\alpha\beta}=\big\langle\mathcal{P}^t,\mathcal{P}^y,\mathcal{I},\mathcal{D}^{\alpha\beta},\mathcal{Z}(f^{\alpha\beta})\big\rangle$$

with $\mathcal{D}^{\alpha\beta}:=(2-\beta)t\partial_t+(x-\alpha)\partial_x+\big((3-\beta)y-\alpha t\big)\partial_y$, $\mathcal{Z}(f^{\alpha\beta}):=f^{\alpha\beta}\partial_u$, and the parameter function $f^{\alpha\beta}=f^{\alpha\beta}(t,x,y)$ running through the solution set of this equation. Modulo the $\mathcal{G}^{\sim}_{\mathcal{F}'}$-equivalence, we can assume $\beta\in(-\infty,5/2]$, and a complete list of $\mathcal{G}^{\sim}_{\mathcal{F}'}$-inequivalent essential Lie symmetry extensions in the class $\mathcal{F}$ is exhausted by the counterparts of those in the class $\mathcal{F}'$, $\mathcal{F}_{00}$ and $\mathcal{F}_{02}$. An analogous list up to the $G^{\sim}_{\mathcal{F}'}$-equivalence consists of the equations $\mathcal{F}_{00}$, $\mathcal{F}_{02}$, $\mathcal{F}_{03}$ and $\mathcal{F}_{05}$.

References

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[3] Vaneeva O.O., Bihlo A. and Popovych R.O., Generalization of the algebraic method of group classification with application to nonlinear wave and elliptic equations, *Commun. Nonlinear Sci. Numer. Simul.* **91** (2020), 105419, arXiv:2002.08939.