The density of Borromean primes

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My talk is concerned with *Arithmetic Topology*, which investigates the interactions between number theory and 3-dimensional topology. The systematic study of this subject was started by B. Mazur, M. Morishita, M. Kapranov and A. Reznikov etc. As one of the analogies in arithmetic topology, the Legendre symbol can be interpreted as Gauss's linking number ([1, Chapter 4]). In [2], Rédei attempted to generalize Gauss's genus theory and introduced a certain triple symbol $[p_1, p_2, p_3]$ for certain primes $p_1,\ p_2, p_3 \equiv 1 \bmod 4$. This symbol may be regarded as a triple generalization of the Legendre symbol $(\frac{p_1}{p_2})$, and it describes the decomposition law of p_3 in a certain dihedral extension over $\mathbb{Q}$ of degree 8, which is determined by $p_1,\ p_2$. Morishita interpreted the Rédei symbol as an arithmetic analogue of Milnor's triple linking number ([1, Chapter 9]). Now *Borromean primes* in the title is defined as arithmetic analogues of Borromean rings:

Definition 1. The triple of primes $\{p_1, p_2, p_3\}$ is called *Borromean primes* when it satisfies the following conditions:

$$p_i \equiv 1 \bmod 4\ (i = 1, 2, 3), \quad \left(\frac{p_i}{p_j}\right) = 1\ (1 \leq i \neq j \leq 3) \quad \text{and} \quad [p_1, p_2, p_3] = -1.$$

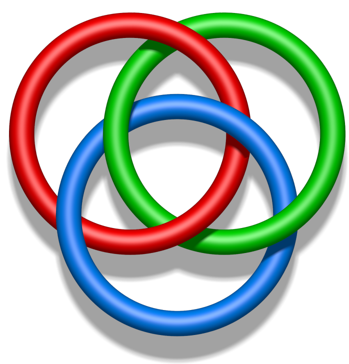

Figure 1.1. Borromean rings

The study of asymptotic distribution of primes goes back to Gauss, and it is viewed as an origin of the so called arithmetic statistics nowadays. Gauss predicted Prime Number Theorem and it was proved independently by Hadamard and de la Vallée Poussin. In 19th century, the following formula was shown by Dirichlet: for coprime integers $m(> 1)$ and a,

$$\pi(x; a, m) \sim \frac{1}{\varphi(m)} \cdot \frac{x}{\log x},$$

where $\pi(x; a, m)$ stands for the number of primes less than or equal to x which have the form $a + km$, and $\varphi(m)$ is the Euler function. It was generalized to the following more general density theorem, known as the Chebotarev density theorem: Let M be a number field and M' be a finite Galois extension of M. For $\sigma \in \mathrm{Gal}(M'/M)$ and any positive real number x, we define

$$\pi_{M'/M}(x; \sigma) := \# \left\{ \mathfrak{p} \in S_M^0 \,\middle|\, \mathfrak{p}\text{: unramified in } M',\ \mathrm{N}_M \mathfrak{p} \leq x,\ \left[\frac{M'/M}{\mathfrak{p}}\right] = C(\sigma) \right\},$$

where S_M^0, $\mathrm{N}_M\mathfrak{p}$, $[\frac{M'/M}{\mathfrak{p}}]$, and $C(\sigma)$ mean the set of prime ideals of M, the absolute norm of $\mathfrak{p} \in S_M^0$, the Artin symbol for $\mathfrak{p} \in S_M^0$ and the conjugacy class of σ in $\mathrm{Gal}(M'/M)$, respectively. Then the Chebotarev density theorem asserts

$$\pi_{M'/M}(x;\sigma) \sim \frac{\#C(\sigma)}{\#G} \cdot \frac{x}{\log x}.$$

Note that Dirichlet's theorem is interpreted as a special case of Chebotarev's theorem in the m-th cyclotomic field $\mathbb{Q}(\zeta_m)$.

In this talk, I will show the density of Borromean primes. Let $\pi_{\mathrm{Borr}}(x)$ denote the number of Borromean primes $\{p_1, p_2, p_3\}$ with $p_i \leq x$ for $i = 1, 2, 3$. Then our main theorem is stated as follows:

Theorem 2 (Main Theorem [3]). *If GRH is true, we have*

$$\lim_{x \to +\infty} \frac{\pi_{\mathrm{Borr}}(x)}{\#\{\{p_1, p_2, p_3\} \mid p_i \leq x \,(\text{for } i = 1, 2, 3),\, p_i \neq p_j \,(i \neq j)\}} = \frac{1}{128},$$

where GRH means Generalized Riemann Hypothesis.

Note that our theorem is not obtained by a straightforward application of Chebotarev density theorems. We need for the proof more elaborate analysis on the error term of the Chebotarev density theorem under GRH ([4]).

References

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[3] Y. Ishida, A. Kuramoto, and D. Zheng. The Density of Borromean Primes. to appear in *Commentarii Mathematici Universitatis Sancti Pauli*.

[4] Loïc Grenié, Giuseppe Molteni. An explicit Chebotarev density theorem under GRH. *Journal of Number Theory*, 200: 441–485, 2019.