Ordinary linear differential operators and connections. Application to curvilinear webs

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The framework is real analytic or holomorphic, the field $\mathbb{K}$ denoting $\mathbb{R}$ or $\mathbb{C}$ according to this framework.

We are given two vector bundles E and F, respectively of rank p and q, over a n-dimensional manifold V. We assume $q < p(n+1)$ (the rank of F is smaller than the rank of J^1E). A linear homogeneous differential operator of order one[1] is a linear morphism of vector bundles $D\colon J^1E \to F$. Associated to it is the partial order equation

$$(*) \quad \mathcal{D}s = 0, \text{ where we set, for any section } s \text{ of } E, \mathcal{D}s := D(j^1 s).$$

The restriction $\sigma_1(D) : T^*(V) \otimes E \to F$ of D to the kernel $T^*(V) \otimes E$ of the projection $J^1E \to E$ is called the principal symbol of D.

More generally, for any integer h (≥ 1), we define by successive derivations the $(h-1)^{th}$ prolongation $D_h : J^hE \to J^{h-1}F$ of D : the solutions of $(*)$ are still the sections s of E such that $D_h(j^h s) = 0$. Recalling, the exact sequence

$$0 \to S^hT^*(V) \otimes E \to J^hE \to J^{h-1}E \to 0,$$

where $S^hT^*(V)$ denotes the h^{th} symetric power of the bundle $T^*(V)$ of 1-forms, we call *principal symbol* of D_h the restriction

$$\sigma_h(D) : S^hT^*(V) \otimes E \to J^{h-1}F$$

of D_h to the sub-bundle $S^hT^*(V) \otimes E$ of J^hE.

We denote by

$$c(n,h) := \frac{(n-1+h)!}{(n-1)!\ h!}$$

the dimension of the $\mathbb{K}$-vector space of homogeneous polynomials of degree h with n unknowns and coefficients in $\mathbb{K}$, which is also the rank of $S^hT^*(V)$, or the number of multi-indices $I = (i_1, ..., i_n)$ of partial derivatives of order $|I| = h$ with respect to n unknowns, (where $|I| = i_1 + i_2 + ... + i_n$).

Definition 1. The differential operator D is said to be ordinary if $q \leq pn$ and, for any h $(h \geq 1)$, the principal symbol $\sigma_h(D)$ has maximal rank $(inf(q.c(n,h-1), p.c(n,h)))$

If D is ordinary, the kernel R_h of D_h is a vector bundle, which is the set of the formal solutions of $(*)$ at order h $\big($with $R_{h+1} = J^1R_h \cap J^{h+1}E$, the intersection being in $J^1(J^hE)\big)$.

Definition 2. The differential operator D is said to be calibrated if $\frac{p(n-1)}{q-p}$ is an integer, strictly positive.

This implies in particular $p < q$.

If $p < q$, we have only finitely many conditions to check for D to be ordinary. In fact, set :

$$h_0 := \Big[\frac{p(n-1)}{q-p}\Big], \quad \text{(the integral part of } \tfrac{p(n-1)}{q-p}).$$

We then get:

[1] The theory works also for differential operators of higher order.

Proposition 3. *For D to be ordinary when $p < q \leq p.n$, it is sufficient that the principal symbols $\sigma_h(D)$ have their rank maximal for $1 \leq h \leq h_0 + 1$ in general (resp. for $1 \leq h \leq h_0$ if D is moreover calibrated).*

We first prove:

Theorem 4. *If D is ordinary and $p < q \leq p.n$, the dimension of the vector space $\mathcal{S}_m$ of germs of solutions of the equation $\mathcal{D}s = 0$ at a point m of V is upper-bounded by the number*

$$\sum_{h=0}^{h_0} \binom{n-1+h}{h} . \frac{p(n-1)-(q-p)h}{n-1+h} \quad \Big(= p.c(n+1, h_0) - q.c(n+1, h_0 - 1)\Big).$$

If D is moreover, calibrated, we define a tautological connection on the vector bundle $\mathcal{E} := R_{h_0-1}$, such that

Theorem 5. *The space $\mathcal{S}$ of solutions of $(*)$ is isomorphic to the space of sections σ of $\mathcal{E}$ whose covariant derivative $\nabla\sigma$ vanishes. Hence, the dimension of $\mathcal{S}_m$ is maximal iff the curvature of this connection vanishes.*

We then apply these results by building, for any curvilinear d-web on V $(d > n)$, a linear differential operator which is *always ordinary and calibrated*, and for which *solutions of $(*)$ are the $(n-1)$-abelian relations* of the web. After theorem 1, we recover the upper-bound already given by Damiano ([3]), for the rank of the web (dimension of the space of $(n-1)$-abelian relations), by taking $p = d - n$ and $q = d - 1$ (hence $h_0 = d - n$).

As a corollary of Theorem 5, we can define for any curvilinear web a notion of "curvature", and prove :

Theorem 6. *The Damiano's upper-bound for the rank of a curvilinear web is reached iff its curvature vanishes.*

The main interest of this result is that it is no more necessary to exhibit the abelian relations for proving the maximality of the rank.

Such a definition for the curvature of a web, whose vanishing is equivalent to the maximality of the rank, goes back to Blaschke ([1]) in the case $n = 2, d = 3$. Various generalizations have been done since, mainly for planar webs ([6, 8, 9]), for webs of codimension one ([2, 4]) when they are "ordinary", and for d-curvilinear webs when $d = n + 1$ ([5]). The definition that we give below for d-curvilinear webs whatever be d is new ; it has been announced in a preprint ([7]).

As an example, we prove (using a computation with Maple) that *the curvature of the exceptional 6-web $W_{0,6}$ in dimension 3 vanishes*, hence recovering that it has has a maximal rank (as well as his 4 and 5-subwebs). In fact all $n+3$-webs $W_{0,n+3}$ (that we re-defined here in words of vector fields) have a maximal rank : this has been claimed by Damiano ([3]) for any n, and proved by him for n even. He made a mistake in the proof for n odd, which has been corrected by Pirio ([10]).

References

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