Classification of smooth structures on line with two origins

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We classified differentiable structures on a line $\mathbf{L}$ with two origins begin a non-Hausdorff but T_1 one-dimensional manifold obtained by "doubling" 0.

Definition 1. Let τ be the standard topology on $\mathbb{R}$. Then $\mathbf{L} = \mathbb{R} \sqcup \bar{0}$ is a disjoint union of $\mathbb{R}$ with some point $\bar{0}$ endowed with the follwoing topology:

$$\eta = \tau \cup \{(W \setminus 0) \cup \{\bar{0}\} : 0 \in W \in \tau\}$$

whose elements are elements of τ and also open neighborhoods of 0 in which 0 is replaced with $\bar{0}$.

For $k \in \mathbb{N} \cup \{\infty\}$ let H^k be the group of homeomorphisms h of $\mathbb{R}$ such that $h(0) = 0$ and the restriction of $h\,\mathbb{R}\backslash 0$ is a $\mathcal{C}^k$-diffeomorphism. It contains a subgroup D^k consisting of $\mathcal{C}^k$-diffeomorphisms of $\mathbb{R}$ also fixing 0.

Definition 2. Let H be a group and C, D be two subgroups. Then for each $h \in H$ the following subset of H:

$$ChD = \{chd^{-1} : c \in C, d \in D\}$$

is called the (C, D)-coset of h. If $C = D$, then DhD is also called the D-*double coset* of h. The set

$$Dh^{\pm 1}D := DhD \cup Dh^{-1}D = \{chd^{-1}, ch^{-1}d^{-1} : c, d \in D\}$$

is called the $(D, \pm)$-*double coset of* h.

We are refering to the book by J. Lee [1] and paper of F. Takens [2] for the definition of $\mathcal{C}^k$-structures. So the problem of classification of smooth $\mathcal{C}^k$-structures on $\mathbf{L}$ can be stated as follows:

Problem 3. Describe the orbits of the action of the group $\mathcal{H}(\mathbf{L})$ on the set of $\mathcal{C}^k$-structures on M.

It is shown that there is a natural bijection between $\mathcal{C}^k$-structures on $\mathbf{L}$ up to a $\mathcal{C}^k$-diffeomorphism and double coset classes $D^k \setminus H^k / D^k$ which can be regarded as the orbit space of the action $D^k \times D^k$ on H^k by the rule $(a, b)h = ahb^{-1}$.

Theorem 4. *Let* $k \in \mathbb{N} \cup \{\infty\}$*. Then*

- $\mathcal{C}^k$*-structures on* $\mathbf{L}$ *up to a* $\mathcal{C}^k$*=diffeomorphism are in one-to-one correspondence with the set* $\mathcal{D}(\mathbb{R}, 0) \setminus \mathcal{H}^k_0(\mathbb{R})^{\pm 1} / \mathcal{D}(\mathbb{R}, 0)$ *of* $(\mathcal{D}(\mathbb{R}, 0), \pm)$*-double coset classes;*
- *while* $\mathcal{C}^k$*-structures on* $\mathbf{L}$ *up to a* $\mathcal{C}^k$*-diffeomorphism fixing* 0 *and* $\bar{0}$ *are in one-to-one correspondence with the set* $\mathcal{D}(\mathbb{R}, 0) \setminus \mathcal{H}^k_0(\mathbb{R}) / \mathcal{D}(\mathbb{R}, 0)$ *of* $\mathcal{D}(\mathbb{R}, 0)$*-double coset classes.*

References

[1] John M. Lee. Introduction to smooth manifolds, volume 218 of Graduate Texts in Mathematics. Springer, New York, second edition, 2013.

[2] Floris Takens. Characterization of a differentiable structure by its group of diffeomorphisms. Bol. Soc. Brasil. Mat., 10(1):17–25, 1979. doi:10.1007/BF02588337.