About one problem of the Gauss-Kuzmin type

Oleh Makarchuk
(Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine)
E-mail: makolpet@gmail.com

Let $[a_0; a_1, ..., a_n, ...]$ — continued fraction, where $a_0 \in Z_+$, $a_j > 0$ for all $j \in N$. Consider left shift operator

$$T([0; a_1, a_2, ..., a_n, ...]) = [0; a_2, a_3, ..., a_{n+1}, ...].$$

Let $f_n(x) = \lambda(T^{-n}((0; x]))$, where $x \in (0; 1]$, $\lambda(\cdot)$ — Lebesgue measure. The problem of finding

$$f(x) = \lim_{n \to +\infty} f_n(x)$$

for classical continued fractions was posed by Gauss. Kuzmin [1] showed that $f(x) = \log_2(x + 1)$ and

$$|f_n(x) - \log_2(1 + x)| \leq C\beta^{(n^\eta)} \ \ \forall x \in (0; 1]$$

for $\eta = 0,5$ some $C > 0$ and $\beta \in (0; 1)$. Levy [2] showed that it is possible to take $\beta = 0,7$ and $\eta = 1$. Wirsing [4] showed that, for the constant $\gamma \approx 0,3037$

$$\psi(x) = \lim_{n \to +\infty} \frac{f_n(x) - \log_2(1 + x)}{(-\gamma)^n} \ \ \forall x \in (0; 1],$$

where $\psi(x)$ — analytic function.

It is known [3] that for each $t \in [0,5; 1]$ there exists a sequence (b_n) such that $b_n \in \{0,5; 1\}$ for all $n \in N$ and $t = [0; b_1, ..., b_n, ...]$. The last image is called A_2-image. A countable set of numbers $t \in [0,5; 1]$ has two A_2-images.

Theorem 1. *For A_2-image the following conditions are true for some numbers $C_1 > C_2 > 0$ and for each $n \in N$*

$$|f_{n+1}(x_1) - f_{n+1}(x_2)| = |f_n((1+x_1)^{-1}) - f_n((1+x_2)^{-1})| + |f_n((0,5+x_1)^{-1}) - f_n((0,5+x_2)^{-1})| \forall x_1, x_2 \in [0,5; 1];$$

$$C_2|x_2 - x_1| \leq |f_n(x_1) - f_n(x_2)| \leq C_1|x_2 - x_1| \ \ \forall x_1, x_2 \in [0,5; 1], x_1 \leq x_2.$$

References

[1] Kuzmin Rodion. On a problem of Gauss. *Dokl. Akad. Nauk SSSR Ser. A* 375-380, 1928.

[2] Levy Paul. Sur les lois de probabilite dont dependent les quotients complets et incomplets dune fraction continue. *Bull. Soc. Math. France* 57: 178-194, 1929.

[3] Pratsiovytyi Mykola., Chuikov Artem. Continuous distributions whose functions preserve tails of a A-continued fraction representation of numbers. *Random Operators and Stochastic Equations*,27(3): 199–206, 2019.

[4] Wirsing Eduard. On the theorem of Gauss-Kuzmin-Levy and a Frobeniustype theorem for function spaces. *Acta Arithmetica* 24: 506-528, 1974.