Homotopy types of stabilizers of Morse-Bott functions on 3-manifolds

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Let M be a smooth 3-manifold, $\mathcal{D}(M)$ be the group of all C^∞ diffeomorphisms of M. For every smooth function $f : M \to \mathbb{R}$ denote by

$$\mathcal{S}(f) = \{h \in \mathcal{D}(M) \mid f \circ h = f\}$$

the stabilizer of f with respect to the natural action of $\mathcal{D}(M)$ on the space of all C^∞ functions on M. It consists of diffeomorphisms leaving invariant each level set of f. Endow $\mathcal{S}(f)$ with the corresponding strong C^∞ Whitney topology.

Let B be a submanifold of M. Then a *regular neighborhood* of B is a vector bundle $p\colon E \to B$ defined on an open neighborhood E of B in M and being a smooth retraction onto B. In that case a function $g\colon E \to \mathbb{R}$ is called 2-*homogeneous* if $g(tx) = t^2 g(x)$ for all $x \in E$ and $t \geq 0$.

Definition 1. Say that a Morse-Bott function $f : M \to \mathbb{R}$ is 2-*homogeneous* if for every critical submanifold B of f of dimension 1 and 2 there exists a tubular neighborhood $p\colon E \to B$ and a 2-homogeneous on fibers function $g\colon E \to \mathbb{R}$ such that $f = g$ near B.

Notice that in general (due to Morse-Bott lemma) f is 2-homogeneous only locally at each critical point x of f.

Now let $f\colon M \to \mathbb{R}$ be a C^∞ Morse-Bott function taking constant values at boundary components of M. Let also Γ be the Kronrod-Reeb graph of f, being the quotient of M by the partition into connected component of every level set of f, and $p : M \to \Gamma$ be the natural projection.

Say that an edge e of Γ is *internal* if its vertices have degrees ≥ 2, i.e. they correspond to non-extremal critical submanifodls of f. At each edge e of Γ fix a point x_e and put $N_e = p^{-1}(x_e)$. Thus, N_e is a closed subsurface of M on which f takes a constant value.

Theorem 2. *Let $f\colon M \to \mathbb{R}$ be a 2-homogeneous Morse-Bott function. Let also n be the total number of those N_e for which*

- *the edge e is internal and*
- *N_e is a 2-sphere or a projective plane.*

Then the higher homotopy groups of $\mathcal{S}(f)$ are n-powers of the corresponding 1-times higher homotopy groups of 2-sphere:

$$\pi_k \mathcal{S}(f) = \underbrace{\pi_{k+1} S^2 \times \cdots \times \pi_{k+1} S^2}_{n}, \quad k \geq 2.$$

In particular, if there are no such spheres and projective spaces, then $\mathcal{S}(f)$ is aspherical.