Deformation properties of smooth functions on Klein bottle

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Let M be a connected compact C^∞-smooth 2-manifold. If $X \subset M$ is a closed subset of M, then $\mathcal{D}(M, X)$ denotes the group of diffeomorphisms of M, which are identity on X, endowed with the strong Whitney topology. If $X = \varnothing$, we omit X from notation. K denotes Klein bottle.

Consider space $C^\infty(M, \mathbb{R})$ endowed with the strong Whitney topology. Then the following right action of $\mathcal{D}(M, X)$ on $C^\infty(M, \mathbb{R})$ is defined: $C^\infty(M, \mathbb{R}) \times \mathcal{D}(M, X) \to C^\infty(M, \mathbb{R})$, $(f, h) \longmapsto f \circ h$. For each $f \in C^\infty(M, \mathbb{R})$, let $\mathcal{S}(f, X)$, $\mathcal{O}(f, X)$ be the stabilizer and the orbit of f with respect to that action. Let $\mathcal{D}_{\mathrm{id}}(M, X)$, $\mathcal{S}_{\mathrm{id}}(f, X)$ and $\mathcal{O}_f(f, X)$ be respective connected components of $\mathcal{D}(M, X)$, $\mathcal{S}(f, X)$, $\mathcal{O}(f, X)$ containing denoted subscripts. Also, we use notation $\mathcal{S}'(f, X) = \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(M, X)$.

Proposition 1. *Let $f \in C^\infty(M, \mathbb{R})$ be such that every its germ in every its critical point is C^∞-equivalent to some homogeneous polynomial without multiple factors, and f is constant on the boundary components of M. Then there are three mutually exclusive possibilities:*

(a) *its Kronrod–Reeb graph Γ_f is acyclic, and there exists component α of some critical level set $f^{-1}(a)$ and open disks $D_1, \ldots, D_m$ such that $K \setminus \alpha = \bigsqcup_{i=1}^m D_i$,*

(b) *Γ_f is acyclic, and there exists component β of some regular level set $f^{-1}(b)$ and open Möbius bands M_1, M_2 such that $K \setminus \beta = M_1 \bigsqcup M_2$,*

(c) *Γ_f has a cycle, and there exists component C of some regular level set $f^{-1}(c)$, corresponding to a point on the cycle, and open cylinders $Q_1, \ldots, Q_m$ such that $K \setminus \{h(C) \mid h \in \mathcal{S}(f)\} = \bigsqcup_{i=1}^m Q_i$.*

Theorem 2. *In the case* (b) *of Proposition 1 there is an isomorphism*

$$\pi_1 \mathcal{O}_f(f) \cong \pi_0 \mathcal{S}(f|_{M_1}, \partial M_1) \times \pi_0 \mathcal{S}(f|_{M_2}, \partial M_2).$$

For Mobius band M group $\pi_0 \mathcal{S}(f|_M, \partial M)$ was computed in [1].

Let $C \subset K$ be a closed curve, that corresponds to point on the cycle of Γ_f. Let Q be the cylinder bounded by C and the next curve among $\{C_1 \equiv C, C_2, \ldots, C_m\} = \{h(C) \mid h \in \mathcal{S}(f)\}$. Denote $G = \pi_1 \mathcal{O}(f|_Q, \partial Q)$, and let $G \wr_{m,\gamma} \mathbb{Z}$ be certain type of wreath product depending on γ.

Theorem 3. *In the case* (c) *of Proposition 1 there are two possibilities:*

(i) *either for every $h \in \mathcal{S}(f)$ equality $h(C) = C$ implies that h preserves orientation of C.*
Then there is an isomorphism

$$\pi_1 \mathcal{O}_f(f) \cong G \wr_m \mathbb{Z},$$

and m can be only odd,

(ii) *or there exists $h \in \mathcal{S}(f)$ such that $h(C) = C$ and h changes orientation of C.*
Then exists an automorphism $\gamma \colon G \to G$ with $\gamma^2 = \mathrm{id}$ such that there is an isomorphism

$$\pi_1 \mathcal{O}_f(f) \cong G \wr_{m,\gamma} \mathbb{Z}.$$

Theorem 4. *Consider composition $T^2 \xrightarrow{\pi} K \xrightarrow{f} \mathbb{R}$, where $f \in C^\infty(M, \mathbb{R})$ is the same as in Proposition 1, and π is the orientable double covering of Klein bottle with the torus. Then there are subgroups*

$\pi_0\mathcal{S}'(f) \hookrightarrow \pi_0\mathcal{S}'(f \circ \pi)$ *and* $\pi_1\mathcal{O}_f(f) \hookrightarrow \pi_1\mathcal{O}_{f\circ\pi}(f \circ \pi)$. *Particularly, considering the respective cases of Theorem 3 holds the following:*

(i) $\pi_1\mathcal{O}_f(f) \cong G \wr_m \mathbb{Z} \hookrightarrow G^2 \wr_m \mathbb{Z} \cong \pi_1\mathcal{O}_{f\circ\pi}(f \circ \pi)$,

(ii) $\pi_1\mathcal{O}_f(f) \cong G \wr_{m,\gamma} \mathbb{Z} \hookrightarrow G \wr_{2m} \mathbb{Z} \cong \pi_1\mathcal{O}_{f\circ\pi}(f \circ \pi)$.

References

[1] Kuznietsova I., Maksymenko S. *Deformational symmetries of smooth functions on non-orientable surfaces*, Topol. Methods Nonlinear Anal., to appear, 2024. arXiv:2308.00577