Extending of partial metrics

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A function $p : X^2 \to [0, +\infty)$ is called *a partial metric* on X if for every $x, y, z \in X$ the following conditions

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;
(p_2) $p(x, x) \le p(x, y)$;
(p_3) $p(x, y) = p(y, x)$;
(p_4) $p(x, z) \le p(x, y) + p(y, z) - p(y, y)$.

are true.

For any partial metric $p : X^2 \to [0, +\infty)$ the function $q_p : X^2 \to [0, +\infty)$, $q_p(x, y) = p(x, y) - p(x, x)$, is a quasi-metric on X and the topology of the partial metric space (X, p) is the topology τ_q of the quasi-metric space (X, q_p). Moreover, the function $d_p : X^2 \to [0, +\infty)$, $d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ is a metric on X.

The following theorem was proved by F. Hausdorff in 1930.

Theorem 1. *Let X be a metrizable space, $A \subseteq X$ be a closed subset and $d_A : A^2 \to \mathbb{R}$ be a compatible metric on A. Then there exists a compatible metric $d : X^2 \to \mathbb{R}$ on X such that $d|_{A^2} = d_A$.*

Problem 2. Let X be a partial metrizable space, $A \subseteq X$ be a closed subset and $p_A : A^2 \to \mathbb{R}$ be a compatible partial metric on A. Does there exist a compatible partial metric $p : X^2 \to \mathbb{R}$ on X such that $p|_{A^2} = p_A$?

Proposition 3. *Let (X, p) be a partial metric space. Then the function $f : X \to \mathbb{R}$, $f(x) = p(x, x)$, is an 1-Lipschitz function with respect to the metric d_p.*

Proposition 4. *Let (X, d) be a metric space and $f : X \to [0, +\infty)$ be an 1-Lipschitz function. Then the function $p : X^2 \to \mathbb{R}$,*

$$p(x, y) = \tfrac{1}{2}(d(x, y) + f(x) + f(y)),$$

is a partial metric on X such that $d = d_p$ and $p(x, x) = f(x)$ for every $x \in X$.

Theorem 5. *Let X be a quasi-pseudometrizable space, A be a closed subset of X and $q_A : A^2 \to \mathbb{R}$ ba a compatible bounded quasi-pseudometric on A. Then there exists a compatible quasi-pseudometric q on X such that $q|_{A^2} = q_A$.*

Corollary 6. *Let X be a partial metrizable space, A be a closed subset of X and $p_A : A^2 \to \mathbb{R}$ ba a compatible partial metric on A such that q_{p_A} is bounded. Then there exists a compatible partial metric p on X such that $p|_{A^2} = p_A$.*

Proposition 7. *There exist a quasi-pseudometric space (X, q), a τ_q-closed set $A \subseteq X$ and a quasi-pseudometric p on A such that*

(1) q and p are equivalent on A;
(2) $\tau_r \not\subseteq \tau_q$ for every extension r of p on X.

References

[1] Hausdorff F. *Erweiterung einer stetigen Abbildung*, Fund. Math., **16** (1930), 353-360.
[2] Hušek M. *Extension mappings and pseudometrics*, Extracta Mathematicae, **25** (3) (2010), 277-308.

[3] Matthews S.G. *Partial Metric Topology*, Proc. 8th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci. **728** (1994), 183-197.