# Axiomatic Development of Complexity Theory for Finite Groups

**Chrystopher L. Nehaniv**
(University of Waterloo, Canada)
*E-mail:* cnehaniv@uwaterloo.ca

*"In any field of mathematics, the study of complexity is the first level of sophistication beyond knowing the building blocks."* – John Rhodes

What are the simplest ways to construct a finite group from its atomic constituents? To understand part-whole relations between finite simple groups ('atoms') and the global structure of finite groups, we axiomatize complexity measures on finite groups. From the Jordan-Hölder theorem and Frobenius-Kalužnin-Krasner-Lagrange embedding in an iterated wreath product, any finite group $G$ can be constructed from a unique collection of simple groups, its Jordan-Hölder factors, each with well-defined multiplicities through iterated extension by simple groups. What is the least number of levels needed in such a hierarchical construction if a level is allowed to include several of these atomic pieces? To pose and answer this question rigorously, we give a natural set of hierarchical complexity axioms for finite groups relating to constructability, extension, quotients, and products, and prove these axioms are satisfied by a unique maximal complexity function **cx**. We prove this function is the same as the minimal number of "spans of gems" or "completely reducible groups" (i.e., direct products of simple groups) in a subnormal series with all factors of this type. Hierarchical complexity **cx** is thus effectively computable, and bounded below by all other complexity measures satisfying the axioms, including generalizations of derived length, Fitting height and solvability. Also, the hierarchical complexity of a normal subgroup is bounded above by the complexity of the whole group, although this is not assumed in the axioms and does not follow from the axioms for general (non-maximal) complexity functions satisfying the axioms.

For solvable finite groups, the unique maximal group complexity measure satisfying the axioms on this class agrees with the restriction of the one for all finite groups, and in addition satisfies an embedding axiom - which decidedly cannot be applied in the general case of all finite groups. In both cases, the complexity of a group is bounded above and below by various natural functions. In particular, hierarchical complexity is sharply bounded above by socle length, which yields a canonical decomposition and satisfies all the axioms except the extension axiom. Examples illustrate applications of the bounds and axiomatic methods in determining complexity of groups. We show also that minimal decompositions need not be unique in terms of what components occur nor their ordering. The complexity axioms are also shown to be independent.