On Beloch's curve that appears when solving real cubic with origami

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Origami construction is generally defined by 7 Axioms (re-)founded by Justin–Huzita–Hatori in 1980's (cf. [Jus86, p.40–45], [Lan10],[Ned22]). Among other things, Axiom 6 allows so-called "neusis", which is not allowed in the straightedge and compass construction.

Axiom 1 (Justin–Huzita–Hatori Axioms 6)**.** *Given two points A and B and two lines l_1 and l_2 on a plane, there exists a fold that places A onto l_1 and B onto l_2 at the same time. In other words, it is possible to construct a certain fold line l such that points symmetrical to points A and B with respect to l are placed on the straight lines l_1 and l_2, respectively.*

This Axiom 6 enables to construct all solutions of any given real cubic equation by using a perfect piece of origami. The following method was initially shown by Beloch in 1936 [Bel36], based on Lill's enjoyable idea [Lil67] (see also [Kat99], [Hul11], [NO15]).

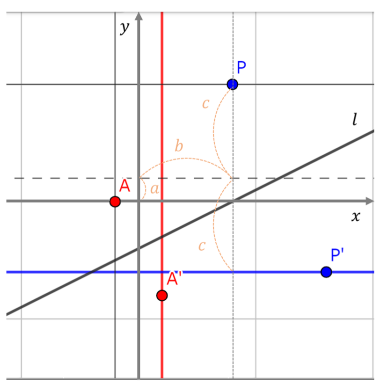
Theorem 2. *Given segments with length $a, b, c \in \mathbb{R}$, every solution of the cubic formula*

$$x^3 - ax^2 - bx + c = 0 \ \cdots \ (*)$$

may be constructed. More precisely,

(1) If we make a fold that places (I) $A(-1, 0)$ onto $x = 1$ and (II) $P(b, a + c)$ onto $y = a - c$, then the y-intercept r of the fold line l is a solution of $(*)$

(2) For any solution $x = r$ of $(*)$, a fold that places $A(-1, 0)$ onto $A'(1, 2r)$ satisfies the conditions (I) and (II) in (1).



In other words, *there is a bijective correspondence between all fold lines satisfying the conditions* (I) and (II) *and all real solutions of* $(*)$. In addition, by considering all fold lines satisfying (I) and parametrizing them by r, we may find all solutions.

We consider all fold lines satisfying the condition (I), and investigate the orbit of the points $P'(r)$ ($r \in \mathbb{R}$), each of which is symmetric to P with respect to a fold line. We set $(p, q) = (b, a + c)$. Our main results may be summarized into the following theorem.

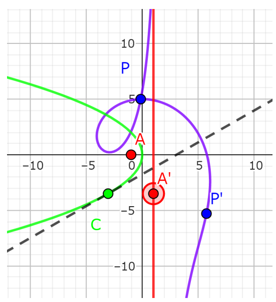
Theorem 3. *The union $\mathcal{F}$ of the orbit of the points P' and the point $P(p, q)$ is a real cubic curve*

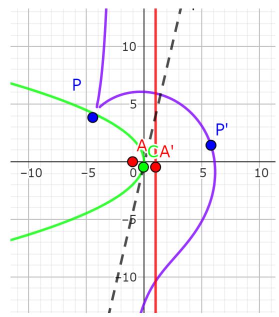
$$F(x, y) := 2(q - y)^2 - (q + y)(q - y)(p - x) - (p - x)^2(p + x) = 0,$$

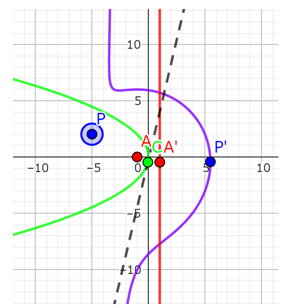
which we call Beloch's curve.

The point P is its uniquely existing singular point, and the Hessian at P is given by $\mathcal{H}_{\mathcal{F}} = -4(4p+q^2)$. We have the following equivalence on the shape of $\mathcal{F}$ and the parabola $\mathcal{G} : 4x + y^2 = 0$.

- *P is on the left side of $\mathcal{G}$ $\iff$ The orbit of P''s does not pass through P $\iff$ P is an isolated point of $\mathcal{F}$.*
- *P is on $\mathcal{G}$ $\iff$ The orbit of P''s passes through P just once $\iff$ P is a cusp of $\mathcal{F}$.*
- *P is on the right side of $\mathcal{G}$ $\iff$ The orbit of P''s passes through P twice $\iff$ P is a self-intersection point of $\mathcal{F}$.*







We may also classify the shapes of real cubic curves $a_0y^2 - a_1xy^2 - a_3x^2 - a_4x^3 = 0$ in a similar manner. In addition, we show that the rotation number of $\mathcal{F}$ around the point A is determined by the relationship between P and $x = 1$.

In the proof of 3, we use the fact that the fold line l is the tangent line of the parabola $\mathcal{G}$ at $(-r^2, 2r)$. In general, given a point A, Q, and a line m, then Axiom 5 allows a fold l such that $Q \in l$ and l maps A onto m. 3 would shed new light on the relationship between Axioms 5 and 6.

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