Magnetic trajectories on 2-step nilmanifolds

Gabriela P. Ovando
(Depto de Matemática, ECEN-FCEIA, Universidad Nacional de Rosario. Pellegrini 250. 2000 Rosario. Argentina)
E-mail: gabriela@fceia.unr.edu.ar

From the mechanical perspective, the behaviour of a charged particle in presence of a force, known as Lorentz force, is described by an equation of the form:

$$\nabla_{\gamma'}\gamma' + F\gamma',$$

where γ is a differentiable curve on Riemannian manifold (M, g), ∇ is the corresponding Levi-Civita connection and F is a skew-symmetric $(1,1)$-tensor such that the corresponding 2-form $g(F\cdot,\cdot)$ is closed. Geodesics are the corresponding curves whenever $F \equiv 0$, that means that the particles do not experience any force. The magnetic trajectories are quite different from geodesics. For instance in the Euclidean plane, while geodesics are straight lines, magnetic trajectories are circles.

In this work, we concentrate the attention to magnetic trajectories associated to a left-invariant Lorentz force on a given 2-step nilpotent Lie group equipped with a left-invariant metric $(N, \langle\ ,\ \rangle)$ and the associated compact quotients. The main purposes are:

(i) to describe the solutions of the magnetic equation;
(ii) to determine closedness conditions of magnetic trajectories on compact quotients.

To facilitate the description of magnetic trajectories through the identity element, we make use of the structure of the Lie algebra. Any 2-step nilpotent Lie algebra with a metric admits an orthogonal decomposition

$$\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}, \qquad \mathfrak{v} = \mathfrak{z}^{\perp},$$

where $\mathfrak{z}$ denotes the center of $\mathfrak{n}$.

One proves that magnetic trajectories for left-invariant Lorentz forces preserving the decomposition above can be explicitly computed, see [1]. In other cases, the magnetic curves obey different features.

We shall show examples of magnetic trajectories. In particular on the Heisenberg Lie group of dimension three, one has examples related to elliptic integrals. As usual, once one finds the curves on the Lie group, one may induce them to compact quotients. In this situation, one search for closed magnetic trajectories.

On the other hand, we discuss obstructions to the existence of (left-invariant) Lorentz forces on 2-step nilpotent Lie algebras.

References

[1] G. Ovando, M. Subils. Magnetic Trajectories on 2-Step Nilmanifolds. *J. Geom. Anaysis*, 33, Art. 186, 2023.
[2] G. Ovando, M. Subils. Magnetic fields on non-singular 2-step nilpotent Lie groups, *J. Pure and Appl. Algebra*, 228 (6), Art. 107609, 2024.