N-FOCI BALLS IN HYPERBOLIC GEOMETRY

Illia Ovtsynov
(Taras Shevchenko National University of Kyiv, Kyiv, Ukraine)
E-mail: iliarkov@gmail.com

Let's suppose that $\mathbb{H}^2 = \{(x, y) \mid y > 0\}$ is an upper half-plane with the Riemannian metric $\dfrac{dx^2 + dy^2}{y^2}$. It is called a hyperbolic plane and has a constant negative Gaussian curvature -1. Besides, $\mathbb{H}^2$ is a Hadamard space, which is a complete Riemannian manifold of nonpositive sectional curvature.

Between two any points $x, y \in \mathbb{H}^2$ there is a unique geodesic $\sigma_{x,y}$. So we can define a notion of a *geodesically convex* (or just *convex*) set in hyperbolic plane — it is a set that for two arbitrary points x and y of its $\sigma_{x,y}$ belongs to this set. Particularly, the mapping

$$\rho : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{R}, \rho(x, y) = \ell(\sigma_{x,y}), x, y \in \mathbb{H}^2,$$

where ℓ denotes a length of curve in $\mathbb{H}^2$, satisfies all the axioms of metric space.

We also can define a notion of convex function in $\mathbb{H}^2$.

Definition 1. We will call a parametrization $\gamma : [0, 1] \to \mathbb{H}^2$ of the geodesics between points a and b in $\mathbb{H}^2$, $\gamma(0) = a$, $\gamma(1) = b$, *standard*, if for all $\alpha \in (0; 1)$ the equality

$$\rho(a, \gamma(\alpha)) = \alpha\ell$$

holds. Here ℓ denotes a length of the appropriate geodesics.

Definition 2. A function $f : \mathbb{H}^2 \to \mathbb{R}$ is called *convex* in a convex set $A \subset \mathbb{H}^2$, if for arbitrary points $x_1, x_2 \in A$ and a standard parametrization $\gamma : [0, 1] \to \mathbb{H}^2$ of the geodesics between them, $\gamma(0) = x_2$, $\gamma(1) = x_1$, next inequality holds:

$$\forall\alpha \in [0, 1] : f(\gamma(\alpha)) \le \alpha f(x_1) + (1 - \alpha)f(x_2). \tag{1}$$

Definition 3. Let's fix in $\mathbb{H}^2$ any mutually distinct points $x_1, \dots, x_N$, where $N \in \mathbb{N}$, and such positive numbers $w_1, \dots, w_N, a$ that $\sum\limits_{k=1}^{N} w_k = 1$. *Open weighted N-foci ball*, or *weighted N-foci ball*, is a set

$$A = \{x \in \mathbb{H}^2 \mid w_1\, \rho(x, x_1) + \dots + w_N\, \rho(x, x_N) < a\}, \tag{2}$$

where $x_1, \dots, x_N$ are called *foci of the weighted N-foci ball*, a is called *a radius of the weighted N-foci ball*, $w_1, \dots, w_N$ are called *weights of the foci $x_1, \dots, x_N$*.

We can define closed weighted N-foci balls the same way, having replaced the symbol "$<$" by the symbol "$\le$" in the formula (2).

Let's fix any point $x_0 \in \mathbb{H}^2$ and define the distance function for it:

$$f : \mathbb{H}^2 \to \mathbb{R},\ f(x) = \rho(x, x_0),\ x \in \mathbb{H}^2.$$

Theorem 4. *The distance function f is convex in the hyperbolic plane $\mathbb{H}^2$.*

It is known, that such a function is convex in any Hadamard space [2]. In this work we got a direct proof of convexity of f for the case of the hyperbolic plane.

From the convexity of f we obtain another result.

Theorem 5. *All open and closed weighted N-foci balls are geodesically convex sets in the hyperbolic plane $\mathbb{H}^2$.*

We also proved geodesical convexity of 1-foci ball, which is a hyperbolic ball, with geometrical methods.

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