A retraction from the space of pseudometrics to the space of ultrapseudometrics

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Definition 1. A **pseudometric** on a set X is a function $d : X \times X \to \mathbb{R}$ that satisfies the following properties for all $x, y, z \in X$:

(1) **Non-negativity**: $d(x, y) \geq 0$.
(2) **Identity of indiscernibles**: $d(x, x) = 0$. (However, it is not required that $d(x, y) = 0$ implies $x = y$, which differentiates a pseudometric from a metric.)
(3) **Symmetry**: $d(x, y) = d(y, x)$.
(4) **Triangle inequality**: $d(x, z) \leq d(x, y) + d(y, z)$.

An **ultrapseudometric** is a type of distance function defined on a set that generalizes the notion of a metric, incorporating properties specific to ultrametrics and pseudometrics. Formally:

Definition 2. An ultrapseudometric d on a set X is a function $d : X \times X \to \mathbb{R}$ that satisfies the above properties of **non-negativity, identity of indiscernibles, and symmetry**, but the **triangle inequality** is satisfied in a stronger form:

(4) **Strong triangle inequality (Ultrametric inequality)**[1] : for all $x, y, z \in X$

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}$$

We denote with $\mathcal{P}\!\int(X)$ the set of all pseudometrics on a fixed set X, and $\mathcal{UP}\!\int(X)$ is its subset consisting of all ultrapseudometrics on X.

Theorem 3. *There is a non-expanding w.r.t. the uniform convergence metric retraction* $\mathcal{P}\!\int(X) \to \mathcal{UP}\!\int(X)$

We rely on the following lemmas.

Lemma 4. *For each pseudometric* $d : X \times X \to \mathbb{R}$ *and a subset* $A \subset X$ *the function* $d_A : X \times X \to \mathbb{R}$ *with the formula*

$$d_A(x, y) = \begin{cases} 0, & x, y \in A \text{ or } x, y \notin A, \\ d(A, X \setminus A), & x \in A, y \notin A \text{ or } x \notin A, y \in A, \end{cases} \qquad x, y \in X,$$

is an ultrapseudometric such that $d_A \leq d$.

Proof. Non-negativity, symmetry, and identity of indiscernibles clearly hold. The only not so trivial part is the strong triangle inequality.[2]

- If all three points are in A or all three are not in A: $d_A(x, z) = 0 \leq \max\{0, 0\}$.
- If $x, y \in A$ and $z \notin A$ (or vice versa): $d_A(x, y) = 0$, $d_A(y, z) = d(A, X \setminus A)$, $d_A(x, z) = d(A, X \setminus A)$, hence $d_A(x, z) \leq \max\{0, d(A, X \setminus A)\}$.
- If $x \in A$, $y \notin A$, and $z \in A$ (or vice versa): $d_A(x, y) = d(A, X \setminus A)$, $d_A(y, z) = d(A, X \setminus A)$, $d_A(x, z) = 0$, hence $d_A(x, z) \leq \max\{d(A, X \setminus A), d(A, X \setminus A)\}$.

To compare d_A and d:

- If $x, y \in A$ or $x, y \notin A$: $d_A(x,y) = 0 \le d(x,y)$.
- If $x \in A$ and $y \notin A$ (or vice versa): $d_A(x,y) = d(A, X \setminus A) \le d(x,y)$.

□

Lemma 5. *For each pseudometric $d : X \times X \to \mathbb{R}$ the function $\bar{d} : X \times X \to \mathbb{R}$ such that*

$$\bar{d}(x,y) = \sup\{d_A(x,y) \mid A \subset X\}, \quad x, y \in X,$$

is the greatest ultrapseudometric on X not exceeeding d.

Proof. (1) **Ultrapseudometric properties of $\overline{d}$:**

(a) **Symmetry:**

$$\overline{d}(x,y) = \sup\{d_A(x,y) \mid A \subseteq X\} = \sup\{d_A(y,x) \mid A \subseteq X\} = \overline{d}(y,x)$$

since d_A is symmetric for all $A \subseteq X$.

(b) **Non-negativity and zero distance:**

$$\overline{d}(x,y) \ge 0$$

and

$$\overline{d}(x,x) = \sup\{d_A(x,x) \mid A \subseteq X\} = 0$$

since $d_A(x,x) = 0$ for all $A \subseteq X$.

(c) **Ultrametric inequality:** For all $x, y, z \in X$:

$$\overline{d}(x,z) = \sup\{d_A(x,z) \mid A \subseteq X\}$$

and

$$\overline{d}(x,z) \le \sup\{\max\{d_A(x,y), d_A(y,z)\} \mid A \subseteq X\} \le \max\{\overline{d}(x,y), \overline{d}(y,z)\}$$

since d_A satisfies the ultrametric inequality for all $A \subseteq X$.

(2) **Comparison $\overline{d} \le d$:** For each $A \subseteq X$, we have $d_A \le d$, thus:

$$\overline{d}(x,y) = \sup\{d_A(x,y) \mid A \subseteq X\} \le d(x,y).$$

(3) **Greatest ultrapseudometric not exceeding d:** Suppose there exists another ultrapseudometric d' on X such that $d' \le d$ and $d' \ge \overline{d}$. Then, for any $A \subseteq X$, $d_A \le d'$, hence:

$$\overline{d} = \sup\{d_A \mid A \subseteq X\} \le d'.$$

Therefore, $\overline{d}(x,y) = \sup\{d_A(x,y) \mid A \subseteq X\}$ is the greatest ultrapseudometric on X not exceeding d. □

We will discuss efficient algorithms for calculation of $\bar{d}$ for a given d on a finite set X.

References

[1] de Groot J. Non-archimedean metrics in topology. *Proc. Amer. Math. Soc.*, 7: 948–953, 1956.
[2] Nykorovych S.I., Nykyforchyn O.R., Zagorodnyuk A.V. Approximation Relations on the Posets of Pseudoultrametrics. *Axioms*, 12(5): 438, 2023.