Action of derivations on polynomials and on Jacobian derivations

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Let $\mathbb{K}$ be an arbitrary field of characteristic zero. Denote by $A := \mathbb{K}[x_1, \ldots, x_n]$ the polynomial ring, and by $R := \mathbb{K}(x_1, \ldots, x_n)$ the field of rational functions in n variables, respectively. A $\mathbb{K}$-linear map $D : A \to A$ is called a $\mathbb{K}$-derivation on A if $D(fg) = D(f)g + fD(g)$ for any $f, g \in A$. The vector space $W_n(\mathbb{K})$ (over $\mathbb{K}$) of all $\mathbb{K}$-derivation is a Lie algebra with respect to the Lie bracket $[D_1, D_2] = D_1D_2 - D_2D_1$, $D_1, D_2 \in W_n(\mathbb{K})$. Recall that every element $D \in W_n(\mathbb{K})$ can be uniquely written in the form

$$D = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}, f_i \in A.$$

The latter means that $W_n(\mathbb{K})$ is a free module of rank n over A with the free generators $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ (see, for example [3], [4]).

Every element D from $W_n(\mathbb{K})$ acts naturally on polynomials from A and on $W_n(\mathbb{K})$ itself (by multiplication). Recall that a polynomial $f \in A$ is a Darboux polynomial for a derivation $D \in W_n(\mathbb{K})$ if $D(f) = \lambda f$ for some $\lambda \in A$, the polynomial λ is called a cofactor for D. One can consider the Darboux polynomials as "eigenvectors" for the derivation D with polynomial "eigenvalues". These (non-constant) polynomials (if they do exist) play significant role in theory of differential equations because for a derivation $D = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}$ one can consider an autonomous system of differential equations of the form

$$\frac{dx_1}{dt} = f_1(x_1, \ldots, x_n), \ldots, \frac{dx_n}{dt} = f_n(x_1, \ldots, x_n)$$

and Darboux polynomials for D are very useful for searching solutions of this system (see, for example, [1], [2]).

We study normalizers of polynomials and derivations under the action of $W_n(\mathbb{K})$ on A and on itself (by multiplication) respectively. For any $f \in A$ one can consider the "normalizer" $N(f)$ in $W_n(\mathbb{K})$ of the form

$$N(f) = \{T \in W_n(\mathbb{K}) \mid T(f) = \lambda f \text{ for some } \lambda \in A\},$$

i.e. $N(f)$ is the set of all the derivations for which f is a Darboux polynomial. This normalizer is a subalgebra of the Lie algebra $W_n(\mathbb{K})$ and it acts on the principal ideal $(f) = Af$ of the ring A. The restriction $\widehat{N}(f)$ of the Lie algebra $N(f)$ on Af is characterized in the next statement.

Theorem 1. *The Lie algebra $\widehat{N}(f)$ is isomorphic to a subalgebra of the semidirect sum $W_n(\mathbb{K}) \leftthreetimes A$.*

Analogously for any $D \in W_n(\mathbb{K})$, one can consider the normalizer of D in $W_n(\mathbb{K})$ of the form

$$N(D) = \{T \in W_n(\mathbb{K}) \mid [T, D] = \lambda D \text{ for some } \lambda \in A\}$$

($N(D)$ is obviously the usual normalizer of the subalgebra AD in the Lie algebra $W_n(\mathbb{K})$). An analogous characterization of $N(D)$ is obtained.

Further, we consider more detailed the Lie algebra $W_2(\mathbb{K})$ and denote for convenience $A = \mathbb{K}[x, y]$. Let $f \in A$, $f \neq 0$. The polynomial f defines a derivation $D_f \in W_2(\mathbb{K})$ by the rule: $D_f(h) = \det J(f, h)$ for any $h \in \mathbb{K}[x, y]$ (here $J(f, h)$ is the Jacobi matrix for f and h). The derivation D_f is called the Jacobian derivation associated with the polynomial f. Note that all the Jacobian derivations form a subalgebra of $W_2(\mathbb{K})$ which coincides with the subalgebra $\mathfrak{sa}_2(\mathbb{K})$ consisting of all divergence-free derivations (see, for example [5]). If for some derivation $T \in W_2(\mathbb{K})$ there exists a Jordan chain consisting of polynomials

$$T(f_1) = \lambda f_1 + f_2, \dots, T(f_{k-1}) = \lambda f_{k-1} + f_k, T(f_k) = \lambda f_k$$

for some $\lambda \in \mathbb{K}$, $k \geq 1$ then we prove the next statement

Theorem 2. *Let $T \in W_2(\mathbb{K})$ acts on polynomials $f_1, \dots f_k$ by the rule*

$$T(f_1) = \lambda f_1 + f_2, \dots, T(f_{k-1}) = \lambda f_{k-1} + f_k, T(f_k) = \lambda f_k$$

for some $\lambda \in \mathbb{K}$, $k \geq 1$. Then the equalities hold:

$$[T, D_{f_1}] = (\lambda - \mathrm{div} T) D_{f_1} + D_{f_2},\ [T, D_{f_2}] = (\lambda - \mathrm{div} T) D_{f_2} + D_{f_3}, \dots,$$
$$[T, D_{f_k}] = (\lambda - \mathrm{div} T) D_{f_k}.$$

The proof of this result is based on the next statement which is of independent interest.

Proposition 3. *Let $T \in W_2(\mathbb{K})$, $f \in \mathbb{K}[x, y]$ and $T(f) = g$ for some polynomial $g \in \mathbb{K}[x, y]$. Then $[T, D_f] = (-\mathrm{div} T) D_f + D_g$. And conversely, if $[T, D_f] = (-\mathrm{div} T) D_f + D_g$ for some $g \in A$, then $T(f) = g + c$ for some $c \in \mathbb{K}$.*

References

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