Periodic point theorem for mappings contracting total pairwise distance

Evgeniy Petrov
(Institute of Applied Mathematics and Mechanics of the NAS of Ukraine, Slovyansk, Ukraine)
E-mail: eugeniy.petrov@gmail.com

We consider a new type of mappings in metric spaces so-called mappings contracting total pairwise distance on n points, see [1]. It is shown that such mappings are continuous. A theorem on the existence of periodic points for such mappings is proved and the classical Banach fixed-point theorem is obtained like a simple corollary as well as the fixed point theorem for mappings contracting perimeters of triangles.

Everywhere below by $|X|$ we denote the cardinality of the set X. Let (X, d) be a metric space, $|X| \geqslant 2$, and let x_1, x_2, ..., $x_n \in X$, $n \geqslant 2$. Denote by

$$S(x_1, x_2, \ldots, x_n) = \sum_{1 \leqslant i < j \leqslant n} d(x_i, x_j)$$

the sum of all pairwise distances between the points from the set $\{x_1, x_2, \ldots, x_n\}$, which we call *total pairwise distance.*

Definition 1. Let $n \geqslant 2$ and let (X, d) be a metric space with $|X| \geqslant n$. We shall say that $T \colon X \to X$ is a *mapping contracting total pairwise distance on n points* if there exists $\alpha \in [0, 1)$ such that the inequality

$$S(Tx_1, Tx_2, \ldots, Tx_n) \leqslant \alpha S(x_1, x_2, \ldots, x_n) \tag{1}$$

holds for all n pairwise distinct points $x_1, x_2, \ldots, x_n \in X$.

Note that the requirement for $x_1, x_2, \ldots, x_n \in X$ to be pairwise distinct is essential, which is confirmed by the following proposition.

Proposition 2. *Suppose that in Definition 1 inequality (1) holds for any n points $x_1, x_2, \ldots, x_n \in X$ with $|\{x_1, x_2, \ldots, x_n\}| = k$, where $2 \leqslant k \leqslant n - 1$. Then T is a mapping contracting total pairwise distance on k points.*

Proposition 3. *Mapping contracting total pairwise distance on m points, $m \geqslant 2$, is a mapping contracting total pairwise distance on n points for all $n > m$.*

Proposition 4. *Mappings contracting total pairwise distance on n points are continuous.*

Let T be a mapping on the metric space X. A point $x \in X$ is called a *periodic point of period n* if $T^n(x) = x$. The least positive integer n for which $T^n(x) = x$ is called the prime period of x. Note that a fixed point is a point of prime period 1.

Theorem 5. *Let $n \geqslant 2$, (X, d) be a complete metric space with $|X| \geqslant n$ and let $T \colon X \to X$ be a mapping contracting total pairwise distance on n points in X. Then T has a periodic point of prime period k, $k \in \{1, \ldots, n-1\}$. The number of periodic points is at most $n - 1$.*

Let (X, d) be a metric space. Then a mapping $T \colon X \to X$ is called a *contraction mapping* on X if there exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leqslant \alpha d(x, y) \tag{2}$$

for all $x, y \in X$.

Corollary 6. *(Banach fixed-point theorem) Let (X, d) be a nonempty complete metric space with a contraction mapping $T \colon X \to X$. Then T admits a unique fixed point.*

The following definition was introduced in [2]. In particular, it is a partial case of Definition 1 when $n = 3$.

Definition 7. Let (X, d) be a metric space with $|X| \geqslant 3$. We shall say that $T\colon X \to X$ is a *mapping contracting perimeters of triangles* on X if there exists $\alpha \in [0, 1)$ such that the inequality

$$d(Tx, Ty) + d(Ty, Tz) + d(Tx, Tz) \leqslant \alpha(d(x, y) + d(y, z) + d(x, z))$$

holds for all three pairwise distinct points $x, y, z \in X$.

The following statement was proved in [2,Theorem 2.4] and it is a direct consequence of Theorem 5 in the case $n = 3$.

Corollary 8. *Let (X, d), $|X| \geqslant 3$, be a complete metric space and let $T\colon X \to X$ be a mapping contracting perimeters of triangles on X. Then T has a fixed point if and only if T does not possess periodic points of prime period 2. The number of fixed points is at most two.*

Proposition 9. *Suppose that under the supposition of Theorem 5 the mapping T has a fixed point x^*, which is a limit of some iteration sequence $x_0, x_1 = Tx_0, x_2 = Tx_1, \ldots$ such that $x_i \neq x^*$ for all $i = 1, 2, \ldots$. Then x^* is the unique fixed point.*

Recall that for a given metric space X, a point $x \in X$ is said to be an *accumulation point* of X if every open ball centered at x contains infinitely many points of X.

Proposition 10. *Let $n \geqslant 2$, (X, d) be a metric space, $|X| \geqslant n$, and let $T\colon X \to X$ be a mapping contracting total pairwise distance on n points. If x is an accumulation point of X, then inequality (2) holds for all points $y \in X$.*

Corollary 11. *Let $n \geqslant 2$, (X, d) be a metric space, $|X| \geqslant n$, and let $T\colon X \to X$ be a mapping contracting total pairwise distance on n points. If all points of X are accumulation points, then T is a contraction mapping.*

References

[1] E. Petrov. Periodic points of mappings contracting total pairwise distance. https://arxiv.org/abs/2402.02536, 2024.

[2] E. Petrov. Fixed point theorem for mappings contracting perimeters of triangles *J. Fixed Point Theory Appl.*, 25, Article No. 74, 2023.