On the semigroup of non-injective monoid endomorphisms of some extension of the bicyclic monoid

Inna Pozdniakova
(Ivan Franko National University of Lviv)
E-mail: inna.pozdniakova@lnu.edu.ua

Oleg Gutik
(Ivan Franko National University of Lviv)
E-mail: oleg.gutik@lnu.edu.ua

Let $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ be the semigroup defined in [1] with the two-element family $\mathscr{F}$ of inductive subset of ω. Without loss of generality we may assume that $\mathscr{F}=\{[0,\infty),[1,\infty)\}$.

In the paper [2] we study injective endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ with the two-elements family $\mathscr{F}$ of inductive nonempty subsets of ω. We describe the elements of the semigroup $\boldsymbol{End}_{*}^{1}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ of all injective monoid endomorphisms of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$.

This work is a continuation of [2].

Fix an arbitrary non-negative integer k. For all $i,j\in\omega$ we define transformations γ_k and δ_k of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ in the following way

$$(i,j,[0,\infty))\gamma_k=(i,j,[1,\infty))\gamma_k=(ki,kj,[0,\infty)).$$

$$(i,j,[0,\infty))\delta_k=(ki,kj,[0,\infty)) \qquad \text{and}$$

$$(i,j,[1,\infty))\delta_k=(k(i+1),k(j+1),[0,\infty))$$

By $\boldsymbol{End}_{0}^{*}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$ we denote the semigroup of all non-injective monoid endomorphisms of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ for the family $\mathscr{F}=\{[0,\infty),[1,\infty)\}$.

Theorems 1 and 2 describe the algebraic structure of the semigroup $\boldsymbol{End}_{0}^{*}(\boldsymbol{B}_{\omega}^{\mathscr{F}})$.

Theorem 1. *If $\mathscr{F}=\{[0,\infty),[1,\infty)\}$, then for any non-injective monoid endomorphism $\mathfrak{e}$ of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ only one of the following conditions holds:*

(1) $\mathfrak{e}$ is the annihilating endomorphism, i.e., $\mathfrak{e}=\gamma_0=\delta_0$;
(2) $\mathfrak{e}=\gamma_k$ for some positive integer k;
(3) $\mathfrak{e}=\delta_k$ for some positive integer k.

Theorem 2. *Let $\mathscr{F}=\{[0,\infty),[1,\infty)\}$. Then for all positive integers k_1 and k_2 the following conditions hold:*

(1) $\gamma_{k_1}\gamma_{k_2}=\gamma_{k_1k_2}$;
(2) $\gamma_{k_1}\delta_{k_2}=\gamma_{k_1k_2}$;
(3) $\delta_{k_1}\gamma_{k_2}=\delta_{k_1k_2}$;
(4) $\delta_{k_1}\delta_{k_2}=\delta_{k_1k_2}$.

References

[1] O. Gutik, M. Mykhalenych. On some generalization of the bicyclic monoi, Visnyk Lviv. Univ. Ser. Mech.–Mat., 90, 2020, p. 5–19 (in Ukrainian).

[2] O. Gutik, I. Pozdniakova. On the semigroup of injective monoid endomorphisms of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ with the two-elements family $\mathscr{F}$ of inductive nonempty subsets of ω, Visnyk L'viv. Univ. Ser. Mech.-Mat, 94, 2022, p. 32–55.