About square roots of matrices over factorial domains

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Let R be a factorial domain with identity $e \neq 0$. Notations: $\mathrm{R}_{n,m}$ and $\mathrm{R}_{n,m}[\lambda]$ are sets of $(n \times m)$ matrices over the domain R and the polynomial ring $\mathrm{R}[\lambda]$ respectively, 0_n and I_n are the zero and the identity $n \times n$ matrices respectively, $\mathbb{C}$ is the field of complex numbers, $\mathbb{R}$ is the field of real numbers and $\mathbb{Z}$ is the ring of integers.

It is said that an $n \times n$ matrix B is a square root of the matrix $A \in \mathrm{R}_{n,n}$ if $B^2 = A$. The computation of matrix square roots arise in a variety of application domains, including in physics, signal processing, optimal control theory, and many others. The problem of finding square roots from a matrix A over $\mathbb{C}$ or $\mathbb{R}$ is well studied (see [1]–[10] and references therein). Unlike square roots of the complex numbers $\mathbb{C}$, the square root of a matrix over $\mathbb{C}$ may not exist. For example, the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}_{2,2}$ has no square roots. However, for a nonsingular matrix A over $\mathbb{R}$ there always exists a square root over $\mathbb{C}$. We note that the existence of a square root of a matrix A over a field depends on the structure of its elementary divisors corresponding to zero eigenvalues. The structure of square roots over a principal ideal domain of the matrix I_n was written in [14].

It is easy to make sure that for nonsingular matrix $A \in \mathrm{R}_{n,n}$ does not always have a square root over R. For the matrix $A = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix} \in \mathbb{Z}_{2,2}$ there is no square root over $\mathbb{Z}$. However, the matrix $B = \begin{bmatrix} i & 0 \\ -i & i \end{bmatrix}$ over the ring of Gaussian integer $\mathbb{Z}[i]$ is the square root of A. In this report we give conditions under which for a matrix $A \in \mathrm{R}_{n,n}$ there exists a square root over R.

Let R be a factorial domain. For the matrix $A \in \mathrm{R}_{n,n}$ there exists a square root over R if and only if the matrix equation $X^2 = A$ is solvable over R. This equation is solvable if and only if the polynomial matrix $A(\lambda) = I_n\lambda^2 - A$ admits the representation in the form $A(\lambda) = (I_n\lambda - B)(I_n\lambda + B)$, where $B \in \mathrm{R}_{n,n}$. From the last equality we have

$$\det A(\lambda) = a(\lambda) = b(\lambda)\widetilde{b}(\lambda), \tag{1}$$

where $b(\lambda), \widetilde{b}(\lambda) \in \mathrm{R}[\lambda]$ – are monic polynomials of degree n. It is evident that condition (1) is the necessary condition for the existence of a square root of the matrix $A \in \mathrm{R}_{n,n}$.

For the matrix $A \in \mathrm{R}_{n,n}$ and the polynomial $b(\lambda) = \lambda^n + \sum_{i=1}^{n} b_i \lambda^{n-i} \in \mathrm{R}[\lambda]$ (a divisor of the characteristic polynomial of $A(\lambda) = I_n\lambda^2 - A$) we construct the matrices

$$T_A = \begin{bmatrix} \vdots & \vdots \\ O_n & A^3 \\ A^2 & O_n \\ O_n & A^2 \\ A & O_n \\ O_n & A \\ I_n & O_n \\ O_n & I_n \end{bmatrix} \in \mathrm{R}_{n(n+1),2n}, \qquad M_b = \begin{bmatrix} I_n & I_n b_1 & I_n b_2 & \dots & I_n b_{n-1} & I_n b_n \end{bmatrix} \in \mathrm{R}_{n,n(n+1)},$$

$$N_b = \begin{bmatrix} I_n b_1 & I_n b_2 + A & I_n b_3 & \dots & I_n b_{n-1} & I_n b_n & O_n \end{bmatrix} \in \mathrm{R}_{n,(n+1)n}.$$

With matrices T_A, M_b and N_b we associate the $(n \times 2n)$ matrices $M_A = M_b T_A$ and $N_A = N_b T_A$.

In the future, we denote by $d_A(\lambda)$ the g.c.d. minors of $(n-1)$-order of the matrix $A(\lambda)$. By virtue of Theorem 1 in [11], we obtain the following statement.

Proposition 1. *Let a matrix $B \in \mathrm{R}_{n,n}$ be a square root of the matrix $A \in \mathrm{R}_{n,n}$, i.e. $B^2 = A$ and $\det(I_n\lambda - B) = b(\lambda)$. If $\left(b(\lambda), \dfrac{\det A(\lambda)}{b(\lambda)}, d_A(\lambda)\right) = e$, then the square root B is uniquely determined by the characteristic polynomial $b(\lambda)$ for the matrix A.*

The proof of the following statements are based on results of papers [12] and [13].

Theorem 2. *Let $A \in \mathrm{R}_{n,n}$ and let $b(\lambda) = \lambda^n + \sum_{i=1}^n b_i\lambda^{n-i} \in \mathrm{R}[\lambda]$ be a divisor of the characteristic polynomial of the matrix $A(\lambda) = I_n\lambda^2 - A$, i.e. $\det A(\lambda) = b(\lambda)\tilde{b}(\lambda)$. If $\left(b(\lambda), \tilde{b}(\lambda), d_A(\lambda)\right) = e$, then for matrix A there exists a square root B with characteristic polynomial $b(\lambda) = \det(I_n\lambda - B)$ if and only if the equation $XM_A = N_A$ is solvable. If the square root B exists, then matrix B is uniquely determined by its characteristic polynomial $b(\lambda)$.*

Corollary 3. *Let $A \in \mathrm{R}_{n,n}$ and let $b(\lambda) = \lambda^n + \sum_{i=1}^n b_i\lambda^{n-i} \in \mathrm{R}[\lambda]$ be a divisor of the characteristic polynomial of the matrix $A(\lambda) = I_n\lambda^2 - A$, i.e. $\det A(\lambda) = b(\lambda)\tilde{b}(\lambda)$. If $d_A(\lambda) = \mathrm{const}$, then for matrix A there exists a square root B with characteristic polynomial $b(\lambda) = \det(I_n\lambda - B)$ if and only if the equation $XM_A = N_A$ is solvable. If the square root B exists, then matrix B is uniquely determined by its characteristic polynomial $b(\lambda)$.*

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