# Approximation by interpolation trigonometric polynomials on the sets of infinitely differentiable functions

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Denote by $C^{\psi}_{\beta}L_1$ the set of $2\pi$–periodic functions, which for all $x \in \mathbb{R}$ can be represented as convolutions of the form

$$f(x)=\frac{a_0}{2}+\frac{1}{\pi}\int\limits_{-\pi}^{\pi}\Psi_{\beta}(x-t)\varphi(t)dt,\ a_0\in\mathbb{R},\ \varphi\in L_1,\ \varphi\perp 1 \qquad (1)$$

with the generating kernel $\Psi_{\beta}$ of the form

$$\Psi_{\beta}(t)=\sum_{k=1}^{\infty}\psi(k)\cos\big(kt-\frac{\beta\pi}{2}\big),\ \psi(k)>0,\ \beta\in\mathbb{R},$$

such that

$$\sum_{k=1}^{\infty}\psi(k)<\infty.$$

The function $\varphi$ in equality (1) is called as $(\psi,\beta)$–derivative of the function $f$ and is denoted by $f^{\psi}_{\beta}$ $(\varphi(\cdot)=f^{\psi}_{\beta}(\cdot))$ [1].

We study approximation properties of the sets $C^{\psi}_{\beta}L_1$, where we use as approximation aggregate the classical interpolation trigonometric Lagrange polynomials, which are defined by odd number of uniformly distributed interpolation nodes.

For arbitrary function $f(x)$ from $C$ by $\widetilde{S}_{n-1}(f;x)$, $n\in\mathbb{N}$, we will denote the trigonometric polynomial of the order $n-1$, which interpolates $f(x)$ in the nodes $x_k^{(n-1)}=\frac{2k\pi}{2n-1}$, $k\in\mathbb{Z}$, namely, such that

$$\tilde{S}_{n-1}(f;x_k^{(n-1)})=f(x_k^{(n-1)}),\ k=0,1,...,2n-2.$$

The polynomial $\widetilde{S}_{n-1}(f;\cdot)$ is unequivocally defined by mentioned interpolation conditions, is called as Lagrange interpolation polynomial and can be represented in the explicit form via Dirichlet kernel

$$D_{n-1}(t)=\frac{1}{2}+\sum_{k=1}^{n-1}\cos kt=\frac{\sin(n-\frac{1}{2})t}{2\sin\frac{t}{2}}$$

in the following way

$$\tilde{S}_{n-1}(f;x)=\frac{2}{2n-1}\sum_{k=0}^{2n-2}f(x_k^{(n-1)})D_{n-1}(x-x_k^{(n-1)}).$$

Let $\mathcal{T}_{2n-1}$ be the space of all trigonometric polynomials of degree at most $n-1$ and let $E_n(f)_{L_1}$ be the best approximation of the function $f\in L_1$ in the metric of space $L_1$, by the trigonometric polynomials $t_{n-1}$ of degree $n-1$, i.e.,

$$E_n(f)_{L_1}=\inf_{t_{n-1}\in\mathcal{T}_{2n-1}}\|f-t_{n-1}\|_{L_1}.$$

Denote by $\tilde{\rho}_n(f;\cdot)$ the deviation of the function $f\in C$ from its interpolation Lagrange polynomial $\tilde{S}_{n-1}(f;\cdot)$

$$\tilde{\rho}_n(f;x)=f(x)-\tilde{S}_{n-1}(f;x).$$

Let

$$\mathfrak{M}=\left\{\psi\in C[1,\infty)\colon \psi(t)>0, \psi(t_1-2\psi((t_1+t_2)/2)+\psi(t_2)\geq 0\ \forall t_1,t_2\in[1,\infty),\ \lim_{t\to\infty}\psi(t)=0\right\}.$$

We consider for each function $\psi\in\mathfrak{M}$ the following characteristics

$$\alpha(t)=\alpha(\psi;t):=\frac{\psi(t)}{t|\psi'(t)|},\quad \psi'(t):=\psi'(t+0),$$

$$\lambda(t)=\lambda(\psi;t):=\frac{\psi(t)}{|\psi'(t)|}.$$

As it was shown in [2], if $\alpha(t)\downarrow 0$ as $t\to\infty$, then the sets $C^{\psi}_{\beta}L_1$ are the sets of infinitely differentiable functions.

Our aim is to establish the asymptotically best possible interpolation analogues of the Lebesgue type inequalities for the functions $f$ from the sets $C^{\psi}_{\beta}L_1$, $\beta\in\mathbb{R}$, where the upper estimates of the quantities $|\tilde{\rho}_n(f;x)|$, $x\in\mathbb{R}$, are expressed via the best approximations $E_n(f^{\psi}_{\beta})_{L_1}$.

The following theorem takes place.

**Theorem 1.** *Let $\psi\in\mathfrak{M}$ and characteristics $\alpha(t)$ and $\lambda(t)$ satisfy the conditions*

$$\alpha(t)\downarrow 0,\quad t\to\infty,$$

$$\lambda(t)\uparrow \infty,\quad t\to\infty.$$

*Then, for arbitrary function $f\in C^{\psi}_{\beta}L_1$, $\beta\in\mathbb{R}$, in every point $x\in\mathbb{R}$ for all $n\in\mathbb{N}$ such that*

$$\alpha(n)\leq\frac{1}{4},$$

*the following inequality takes place*

$$|\tilde{\rho}_n(f;x)|\leq\frac{2}{\pi}\left|\sin\frac{2n-1}{2}x\right|\psi(n)\lambda(n)\left(1+4\alpha(n)+\frac{1}{\lambda(n)}\right)E_n(f^{\psi}_{\beta})_{L_1}.$$

*Moreover for arbitrary function $f\in C^{\psi}_{\beta}L_1$ one can find the function $\mathcal{F}(\cdot)=\mathcal{F}(f;n;x,\cdot)$ such that $E_n(\mathcal{F}^{\psi}_{\beta})_{L_1}=E_n(f^{\psi}_{\beta})_{L_1}$ and the following equality takes place*

$$|\tilde{\rho}_n(\mathcal{F};x)|=\frac{2}{\pi}\left|\sin\frac{2n-1}{2}x\right|\psi(n)\lambda(n)\left(1+\xi_1\alpha(n)+\frac{\xi_2}{\lambda(n)}\right)E_n(f^{\psi}_{\beta})_{L_1},$$

*where $-4(1+2\pi)\leq\xi_1<\frac{8}{3}(1+\pi)$, $-(1+2\pi)\leq\xi_2\leq 2(1+\pi)$.*

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## References

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