Fundamental solution of non-Archimedean pseudo-differential equation of p-adic argument

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The Vladimirov-Taibleson pseudo-differential operator D^α plays a role of a differential operator in the p-adic analysis (see [1, 3]). The analogue of the wave equation for radial functions in t on non-Archimedean spaces

$$D_t^\alpha u - D_x^{\alpha,n} u = 0 \tag{1}$$

was studied in [2].

In present work the fundamental solution of a more general Cauchy problem for the functions of two p-adic variables, radial in t, was found. The main result is stated in Theorem 2. Theorem 3 proves the uniqueness of the solution in the Lizorkin space of locally constant functions $\Phi(\mathbb{Q}_p^n)$, and Theorem 4 gives an estimate of the norm of the solution in L^1-space.

Let $0 < \alpha < 1$, $\beta > 0$. Consider the eigenvalue problem

$$D^\alpha u = \lambda u,\ \lambda = p^{\beta N},\ N \in \mathbb{Z}, \tag{2}$$

where u is not identically zero.

We also suppose that

$$\beta = K\alpha \text{ for some } K \in \mathbb{N}. \tag{3}$$

Proposition 1. *If the condition* (3) *holds, the equation* (2) *has the set of solutions in* $\Phi(\mathbb{Q}_p)$ *of the following form for* $N \in \mathbb{Z}$*:*

$$u_N(t) = \begin{cases} C_N p^{KN}(1 - \frac{1}{p}), & |t|_p \le p^{-KN}; \\ -C_N p^{KN-1}, & |t|_p = p^{-KN+1}; \\ 0, & |t|_p \ge p^{-KN+2}. \end{cases} \tag{4}$$

Let $0 < \alpha < 1$, $\beta > 0$. We consider the Cauchy problem

$$D_{|t|_p}^\alpha u(|t|_p, x) - D_x^\beta u(|t|_p, x) = 0,\ (t,x) \in \mathbb{Q}_p^+ \times \mathbb{Q}_p^n, \tag{5}$$

$$u(0,x) = u_0(x),\ x \in \mathbb{Q}_p^n, \tag{6}$$

where $u = u(|t|_p, x)$ is a radial function with respect to t, $n \ge 1$.

Theorem 2. *Let* $0 < \alpha < 1$, $\beta > 0$ *such that the condition* (3) *holds. Suppose that the function* u_0 *is in* $\Phi(\mathbb{Q}_p)^n$*. Then the Cauchy problem* (5)*-*(6) *has a solution* $u = u(|t|_p, x)$*, radial in* t*, that belongs to the space* $\Phi(\mathbb{Q}_p^+)$ *for each* $x \in \mathbb{Q}_p$*, and belongs to* $\Phi(\mathbb{Q}_p^n)$ *for each* $t \in \mathbb{Q}_p^+$*.*

If the condition (3) *does not hold, then the equation* (5) *has only a zero solution* $u(t,x) \equiv 0$*,* $t, x \in \mathbb{Q}_p^+ \times \mathbb{Q}_p$*.*

The solution built in the proof of Theorem 2 has the following form

$$u(|t|_p, x) = (\mathcal{F}_{\xi\to x}^{-1}\hat{u})(|t|_p, x) = \Big((\mathcal{F}_{\xi\to x}^{-1} b) * u_0\Big)(|t|_p, x),\ (t,x) \in \mathbb{Q}_p^+ \times \mathbb{Q}_p^n, \tag{7}$$

We consider the problem (5)-(6) in the class of generalized functions, radial in t.

Denote by $\Phi'(\mathbb{Q}_p^+, \Phi'(\mathbb{Q}_p^n))$ the set of distributions over the test function space $\Phi(\mathbb{Q}_p^n)$, with values in $\Phi'(\mathbb{Q}_p^n)$.

Theorem 3. *Let* $F \in \Phi'(\mathbb{Q}_p^+, \Phi'(\mathbb{Q}_p^n))$ *be a generalised solution of the equation* (5)*, that is*

$$\langle\langle F, D_t^\alpha \varphi_1\rangle, \varphi_2\rangle = \langle\langle F, \varphi_1\rangle, D_x^\beta \varphi_2\rangle,$$

for any $\varphi_1 \in \Phi(\mathbb{Q}_p^+)$*,* $\varphi_2 \in \mathbb{Q}_p$*. If* F *is radial in* t*, then* $F \in \mathcal{D}(\mathbb{Q}_p^+, \Phi'(\mathbb{Q}_p^n))$*. If, in addition,* $F(0,x) = 0$*, then* $F(t,x) \equiv 0$*.*

It follows from Theorem 3 that the solutions of the Cauchy problems constructed in Theorem 2 are unique in the class of radial in t, bounded locally constant functions.

Theorem 4. *Suppose that the conditions of Theorem 2 hold. Then the solution of the problem* (5)-(6)*, defined in* (7)*, satisfies the following estimate in* $L^1(\mathbb{Q}_p^n)$ *in variable* x

$$||u(|t|_p, \cdot)||_{L_1(\mathbb{Q}_p^n)} \leq p^{2n\gamma} ||u_0||_{L_1(\mathbb{Q}_p^n)}, \tag{8}$$

where $\gamma \geq \dfrac{2}{K}$ *is a positive constant.*

References

[1] S. Albeverio, A. Yu. Khrennikov, and V. M. Shelkovich, Theory of p-Adic Distributions, Cambridge University Press, 2010.

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[3] V. S. Vladimirov, I. V. Volovich and E. I. Zelenov, *p-Adic Analysis and Mathematical Physics*, World Scientific, Singapore, 1994.