On the semigroup of injective monoid endomorphisms of a some extension of the bicyclic semigroup

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In this paper we shall follow the semigroup terminology of [5].

By ω we denote the set of all non-negative integers.

Let $\mathscr{P}(\omega)$ be the family of all subsets of ω. For any $F \in \mathscr{P}(\omega)$ and any integer n we put $n + F = \{n+k \colon k \in F\}$ if $F \neq \varnothing$ and $n+\varnothing = \varnothing$. A subfamily $\mathscr{F} \subseteq \mathscr{P}(\omega)$ is called *ω-closed* if $F_1 \cap (-n+F_2) \in \mathscr{F}$ for all $n \in \omega$ and $F_1, F_2 \in \mathscr{F}$. For any $a \in \omega$ we denote $[a) = \{x \in \omega \colon x \geqslant a\}$.

On the set $\boldsymbol{B}_\omega = \omega \times \omega$ we define the semigroup operation "$\cdot$" in the following way

$$(i_1, j_1) \cdot (i_2, j_2) = \begin{cases} (i_1 - j_1 + i_2, j_2), & \text{if } j_1 \leqslant i_2; \\ (i_1, j_1 - i_2 + j_2), & \text{if } j_1 \geqslant i_2. \end{cases}$$

It is well known that the bicyclic monoid is isomorphic to the semigroup $\boldsymbol{B}_\omega$.

The following construction is introduced in [1].

Let $\mathscr{F}$ be an ω-closed subfamily of $\mathscr{P}(\omega)$. On the set $\boldsymbol{B}_\omega \times \mathscr{F}$ we define the semigroup operation "$\cdot$" in the following way

$$(i_1, j_1, F_1) \cdot (i_2, j_2, F_2) = \begin{cases} (i_1 - j_1 + i_2, j_2, (j_1 - i_2 + F_1) \cap F_2), & \text{if } j_1 \leqslant i_2; \\ (i_1, j_1 - i_2 + j_2, F_1 \cap (i_2 - j_1 + F_2)), & \text{if } j_1 \geqslant i_2. \end{cases}$$

In [1] is proved that if the family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ is ω-closed then $(\boldsymbol{B}_\omega \times \mathscr{F}, \cdot)$ is a semigroup. Moreover, if an ω-closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ contains the empty set $\varnothing$ then the set $\boldsymbol{I} = \{(i, j, \varnothing) \colon i, j \in \omega\}$ is an ideal of the semigroup $(\boldsymbol{B}_\omega \times \mathscr{F}, \cdot)$. For any ω-closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ the following semigroup

$$\boldsymbol{B}_\omega^{\mathscr{F}} = \begin{cases} (\boldsymbol{B}_\omega \times \mathscr{F}, \cdot)/\boldsymbol{I}, & \text{if } \varnothing \in \mathscr{F}; \\ (\boldsymbol{B}_\omega \times \mathscr{F}, \cdot), & \text{if } \varnothing \notin \mathscr{F} \end{cases}$$

is defined in [1].

In the paper [2] injective endomorphisms of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ with the two-elements family $\mathscr{F}$ of inductive nonempty subsets of ω are studies. Here the authors describe the elements of the semigroup $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ of all injective monoid endomorphisms of the monoid $\boldsymbol{B}_\omega^{\mathscr{F}}$, and show that Green's relations $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, $\mathscr{D}$, and $\mathscr{J}$ on $\boldsymbol{End}_*^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ coincide with the relation of equality. In [3, 4] the semigroup $\boldsymbol{End}^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ of all monoid endomorphisms of the monoid $\boldsymbol{B}_\omega^{\mathscr{F}}$ is studied.

Example 1. Let $\mathscr{F}^3 = \{[0), [1), [2)\}$. Fix an arbitrary positive integer k. We define the transformation $\alpha_{[k]}$ of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}^3}$ in the following way

$$(i, j, [p))\alpha_{[k]} = \begin{cases} (ki, kj, [p)), & \text{if } p \in \{0, 1\}; \\ (k(i+1) - 1, k(j+1) - 1, [2)), & \text{if } p = 2, \end{cases}$$

for all $i, j \in \omega$. It is obvious that $\alpha_{[k]}$ is an injective transformation of the monoid $\boldsymbol{B}_\omega^{\mathscr{F}^3}$.

Lemma 2. *For an arbitrary positive integer k the transformation $\alpha_{[k]} \colon \boldsymbol{B}_\omega^{\mathscr{F}^3} \to \boldsymbol{B}_\omega^{\mathscr{F}^3}$ is an injective monoid endomorphism of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}^3}$.*

Theorem 3. *Let $\mathscr{F}^3 = \{[0), [1), [2)\}$ and ε be an injective monoid endomorphism of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$. Then $\varepsilon = \alpha_{[k]}$ for some positive integer k.*

By $(\mathbb{N}, \cdot)$ we denote the multiplicative semigroup of positive integers.

Theorem 4. *Let $\mathscr{F}^3 = \{[0), [1), [2)\}$. Then the monoid $\boldsymbol{End}_*^1(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})$ of all injective endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ is isomorphic to $(\mathbb{N}, \cdot)$.*

References

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