Subwreath product as structure of normal subgroups of permutational wreath products

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In this research we continue our previous investigation of wreath product normal structure [1, 2] Normal subgroups and there structures for finite and infinite iterated wreath products $S_{n_1} \wr \ldots \wr S_{n_m}$, $n, m \in \mathbb{N}$ and $A_n \wr S_n$ are founded.

Let $k(\pi)$ be the number of cycles in decomposition of permutation π of degree n.

The number $n - k(\pi)$ is denoted by $dec(\pi)$, and is called a decrement [6] of permutation π. As well known [6] the minimal number of transpositions in factorization of a permutation π on transpositions is happen to be equal to $dec(\pi)$. We set $dec(e) = 0$. If $\pi_1, \pi_2 \in S_n$, then the following formula holds:

$$dec(\pi_1 \cdot \pi_2) = dec(\pi_1) + dec(\pi_2) - 2m, m \in \mathbf{N}, \quad (1)$$

Definition 1. The permutational *subwreath product* $G \wr\wr H$ is the semi-direct product $G \ltimes \tilde{H}^X$, where G acts on the subdirect product [4] $\tilde{H}^X$ by the respective permutations of the subdirect factors. Provided the specification of $\tilde{H}^X$ is established separately.

Definition 2. The set of elements from $S_n \wr S_n, n \geqslant 3$ which presented by the tableaux of form: $[e]_0$, $[a_1, a_2, \ldots, a_n]_1$, satisfying the following condition

$$\sum_{i=1}^{n} dec([a_i]_1) = 2k, k \in \mathbb{N}, \quad (2)$$

be called set of type $\widetilde{A}_n^{(1)}$. Note that condition (2) uniquely identifies subdirect product.

The set $\widetilde{A}_n^{(1)}$ is a normal subgroup having **normal rank** 2 in $S_n \wr S_n$ and be denoted by $E \wr\wr \widetilde{A}_n$. We spread this definition on 3-multiple wreath product by recursive way.

Definition 3. *The subgroup $E \wr \widetilde{A}_n^{(1)}$ be denoted by $\widetilde{A}_n^{(2)}$.*

Furthermore we prove that $E \wr \widetilde{A}_n^{(2)} \lhd S_n \wr S_n \wr S_n$. The order of $E \wr \widetilde{A}_n^{(2)}$ is $(n!)^{3n} : 2^3$. The subgroup $\widetilde{A}_n^{(1)}$ has **normal rank** 2 in $S_n \wr S_n$.

Definition 4. The set of elements from $S_n \wr S_n \wr S_n, n \geqslant 3$ presented by the tables [3] form: $[e]_1$, $[e, e, \ldots, e]_2, [a_1, a_2, \ldots, a_n]_2$, satisfying the following condition

$$\sum_{i=1}^{n} dec([a_i]_2) = 2k, k \in \mathbb{N}, \quad (3)$$

be denoted by $\widetilde{A}_{n^2}^{(2)}$. Note that condition (3) uniquely identifies subdirect product in $\prod_{i=1}^{n^2} S_n$ as base of subwreath product, the similar subdirect product describing commutator of wreath product was investigated by us in [8] in research of pronormality it appears in [9].

Proposition 5. *The subgroup $\widetilde{A}_n^{(1)} \lhd S_n \wr S_n$ as well as $\widetilde{A}_n^{(2)} \lhd S_n \wr S_n \wr S_n$. Furthermore $\widetilde{A}_n^{(2)} \lhd \widetilde{A}_{n^2}^{(2)}$.*

Definition 6. *A subgroup in $S_n \wr S_n$ is called $\widetilde{T_n}$ if it consists of:*

(1) elements of $E \wr A_n$,

(2) elements with the tableau [3] *presentation* $[e]_1,\ [\pi_1,\dots,\pi_n]_2$*, that* $\pi_i \in S_n \setminus A_n$*.*

One easy can validates a correctness of this definition, i.e. that the set of such elements form a subgroup and its normality. This subgroup has structure

$$\tilde{T}_n \simeq (\underbrace{A_n \times A_n \times \cdots \times A_n}_{n}) \rtimes C_2 \simeq \underbrace{S_n \boxplus S_n \ldots \boxplus S_n}_{n},$$

where the operation $\boxplus$ *of a subdirect product is subject of item 1) and 2)*

Remark 7. The order of $\widetilde{T_n}$ is $\frac{(n!)^n}{2^{n-1}}$.

Definition 8. The unique minimal normal subgroup is called the monolith.

Theorem 9. *The monolith of* $S_n \wr S_m$ *is* $e \wr A_m$*.*

Theorem 10. *Proper normal subgroups in* $S_n \wr S_m$*, where* $n, m \geq 3$ *with* $n, m \neq 4$ *are of the following types:*

(1) subgroups that act only on the second level are

$$E \wr\wr \widetilde{A_m},\ \widetilde{T_m},\ E \wr S_m, E \wr A_m,$$

(2) subgroups that act on both levels are $A_n \wr\wr \widetilde{A_m},\ S_n \wr\wr \widetilde{A_m},\ A_n \wr S_m,$

wherein the subgroup $S_n \wr\wr \widetilde{A_m} \simeq S_n \leftthreetimes (\underbrace{S_m \boxtimes S_m \boxtimes S_m \ldots \boxtimes S_m}_{n})$ *endowed with the subdirect product satisfying to condition* (2)*.*

Theorem 11. *The full list of normal subgroups of* $W = S_n \wr S_n \wr S_n$ *consists of 50 normal subgroups. These subgroups are the following:*

1 ***Type*** T_{023} *contains:* $E \wr\wr \tilde{A}_n \wr\wr H,\ \ \widetilde{T_n} \wr H$*, where* $H \in \{\tilde{A}_n, \tilde{A}_{n^2}, S_n\}$*. There are 6 subgroups.*

2 ***The second type of subgroups is subclass in*** T_{023} *with new base of wreath product subgroup* $\tilde{A}_{n^2}$*:* $E \wr S_n \wr\wr \tilde{A}_{n^2},\ \ E \wr N_i(S_n \wr S_n)$*. Therefore this class has 12 new subgroups. Thus, the total number of normal subgroups in* ***Type*** T_{023} *is 18.*

3 ***Type*** T_{003}*:* $A^{(3)}_{00(n^2)},\ \widetilde{T_{n^2}},\ \widetilde{T_n}^{(3)}$*.*

4 ***Type*** T_{123}*:* $N_i(S_n \wr S_n) \wr S_n,\ N_i(S_n \wr S_n) \wr \tilde{A}_n$ *and* $N_i(S_n \wr S_n) \wr\wr \tilde{A}_{n^2}$*. Thus, there are 29 new normal subgroups in* T_{123}*, taking into account a repetition.*

Remark 12. Note that $E \wr \widetilde{A}^{(1)}_n \simeq E \wr (E \wr\wr \widetilde{A}_n)$ contains in the family $E \wr N_i(S_n \wr S_n)$.

We denote by $Aut_f X^*$ the group of all finite automorphism of spherically homogeneous rooted tree.

Theorem 13. *Let* $H \lhd Aut_f X^*$ *having depth* k*, then* H *contains* k*-th level subgroup* P *having all even vertex permutations* $p_{ki} \in A_n$ *on* X^k *and trivial permutations in vertices of rest of levels.*

Furthermore P *is normal in* $Aut_f X^*$ *provided* k *is last active level of* $Aut_f X^*$*.*

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