Uniform approximation by Fourier sums in Weyl–Nagy classes $W^r_{\beta,1}$

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We investigate asymptotic behavior of the least upper bounds of approximations in the uniform metric by Fourier sums $S_{n-1}(f;\cdot)$ of classes $W^r_{\beta,1}$ of 2π-periodic Weyl–Nagy differentiable functions f.

Let L_p, $1 \le p \le \infty$, and C be the spaces of 2π–periodic functions with standard norms $\|\cdot\|_{L_p}$ and $\|\cdot\|_C$, respectively.

Further, let $W^r_{\beta,p}$, $r > 0$, $\beta \in \mathbb{R}$, $1 \le p \le \infty$, be classes of 2π-periodic functions f that can be represented in the form of convolution

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int\limits_{-\pi}^{\pi} \varphi(x-t)B_{r,\beta}(t)dt, \quad a_0 \in \mathbb{R}, \tag{1}$$

with Weyl–Nagy kernels of the form $B_{r,\beta}(t) = \sum\limits_{k=1}^{\infty} k^{-r}\cos\Big(kt - \frac{\beta\pi}{2}\Big)$, of function φ satisfying the condition $\varphi \in B^0_p = \Big\{\varphi \in L_p : \|\varphi\|_p \le 1, \int\limits_{-\pi}^{\pi}\varphi(t)dt = 0\Big\}$.

The classes $W^r_{\beta,p}$ are called the Weyl–Nagy classes and the function φ in representation (1) is called the (r,β)-derivative of the function f in the Weyl–Nagy sense and denoted by f^r_β.

Theorem 1. *Let $r > 2$, $\beta \in \mathbb{R}$, and $n \in \mathbb{N}$. The following estimate is true*

$$\mathcal{E}_n(W^r_{\beta,1})_C = \sup_{f \in W^r_{\beta,1}} \|f(\cdot) - S_{n-1}(f;\cdot)\|_C = \frac{1}{n^r}\left(\frac{1}{\pi(1-e^{-r/n})} + \mathcal{O}(1)\delta_{r,n}\right), \tag{2}$$

where $\mathcal{O}(1)$ is a quantity uniformly bounded in all analyzed parameters,

$$\delta_{r,n} = \begin{cases} 1 + \dfrac{n}{r(r-2)}, & 2 < r \le n+1, \\ \dfrac{r}{n^2}e^{-r/n}, & n+1 \le r \le n^2, \\ e^{-r/n} & r \ge n^2. \end{cases}$$

Remark 2. Estimate (2) was published for $r \ge \sqrt{n} + 1$ in [1, 2] (2019, 2022).

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References

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