On the asymptotic behavior of solutions to nonlinear Beltrami equation

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Let $\mathbb{C}$ be the complex plane. In the complex notation $f = u + iv$ and $z = x + iy$, the *Beltrami equation* in a domain $G \subset \mathbb{C}$ has the form

$$f_{\overline{z}} \,=\, \mu(z) f_z, \tag{1}$$

where $\mu\colon G \to \mathbb{C}$ is a measurable function and

$$f_{\overline{z}} = \frac{1}{2}(f_x + i f_y) \qquad \text{and} \qquad f_z = \frac{1}{2}(f_x - i f_y)$$

are formal derivatives of f in $\overline{z}$ and z, while f_x and f_y are partial derivatives of f in the variables x and y, respectively.

We consider the following equation written in the polar coordinates (r, θ)

$$f_\theta = \sigma(re^{i\theta})\, |f_r|^m \, f_r. \tag{2}$$

We rewrite the equation (2) in the Cartesian form:

$$f_{\overline{z}} \,=\, \frac{z}{\overline{z}}\, \frac{1 + i\sigma(z)\, |z|^{-m-1} |z f_z + \overline{z} f_{\overline{z}}|^m}{1 - i\sigma(z)\, |z|^{-m-1} |z f_z + \overline{z} f_{\overline{z}}|^m}\, f_z. \tag{3}$$

Assuming $m = 0$, the equation (3) also becomes the standard linear Beltrami equation (1) with

$$\mu(z) \,=\, \frac{z}{\overline{z}}\, \frac{1 + i\sigma(z)/|z|}{1 - i\sigma(z)/|z|}\ .$$

Choosing $m = 0$ and $\sigma = i|z|$ in (3), we arrive at the classical Cauchy-Riemann system. Later on we assume that $m > 0$.

A mapping $f\colon G \to \mathbb{C}$ is called *regular at a point* $z_0 \in G$, if f has the total differential at this point and its Jacobian $J_f = |f_z|^2 - |f_{\bar{z}}|^2$ does not vanish. A homeomorphism f of Sobolev class $W^{1,1}_{\rm loc}$ is called *regular*, if $J_f > 0$ a.e. By a *regular solution of the equation* (3) we call a regular homeomorphism $f\colon G \to \mathbb{C}$, which satisfies (3) a.e. in G.

Later on, we use the following notations:

$$B_r = \{z \in \mathbb{C} : |z| < r\}\,, \quad \mathbb{B} = \{z \in \mathbb{C} : |z| < 1\}$$

and

$$\gamma_r = \{z \in \mathbb{C} : |z| = r\}.$$

Theorem 1. *Let $f:\mathbb{B}\to\mathbb{C}$ be a regular homeomorphic solution of the equation (3) which belongs to Sobolev class $W^{1,2}_{\rm loc}$, and normalized by $f(0)=0$. Assume that $C>0$ and the coefficient $\sigma:\mathbb{B}\to\mathbb{C}$ satisfies the following condition*

$$\int\limits_{\gamma_r}\frac{|\sigma(z)|^{m+2}}{(\operatorname{Im}\sigma(z))^{m+1}}\,|dz|\leqslant C\,r^2$$

for a.a. $r\in(0,1)$. Then

$$\limsup_{z\to0}\frac{|f(z)|}{|z|}\geqslant\left(\frac{2\pi}{C}\right)^{\frac{1}{m}}.$$

Corollary 2. *Let $f:\mathbb{B}\to\mathbb{C}$ be a regular homeomorphic solution of the equation (3) which belongs to Sobolev class $W^{1,2}_{\rm loc}$, and normalized by $f(0)=0$ and $K>0$. Assume that the coefficient $\sigma:\mathbb{B}\to\mathbb{C}$ satisfies the following condition*

$$\frac{|\sigma(z)|^{m+2}}{(\operatorname{Im}\sigma(z))^{m+1}}\leqslant K\,|z|$$

for a.a. $z\in\mathbb{B}$. Then

$$\limsup_{z\to0}\frac{|f(z)|}{|z|}\geqslant K^{-\frac{1}{m}}.$$

Theorem 3. *Let $f:\mathbb{B}\to\mathbb{C}$ be a regular homeomorphic solution of the equation* (3) *which belongs to Sobolev class $W^{1,2}_{\rm loc}$, and normalized by $f(0)=0$. Suppose that*

$$\sigma_0=\liminf_{\varepsilon\to0}\frac{1}{\pi\varepsilon^2}\int\limits_{B_\varepsilon}\frac{|\sigma(z)|^{m+2}}{|z|\,(\operatorname{Im}\sigma(z))^{m+1}}\,dxdy.$$

1) If $\sigma_0\in(0,\infty)$, then

$$\limsup_{z\to0}\frac{|f(z)|}{|z|}\geqslant c_m\,\sigma_0^{-\frac{1}{m}},$$

where c_m is a positive constant depending on the parameter m.

2) If $\sigma_0=0$, then

$$\limsup_{z\to0}\frac{|f(z)|}{|z|}=\infty.$$

References

[1] I. Petkov, R. Salimov, M. Stefanchuk. Nonlinear Beltrami equation: lower estimates of Schwarz Lemma's type. *Canadian Mathematical Bulletin*, 2023. DOI: https://doi.org/10.4153/S0008439523000942