On the type of Grassman image of a time-like minimal surface in Minkowski space

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In the Minkowski space ${}^1R_4$ there is a coordinate system in which the metric of the space has the form $ds^2 = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$. Let the equation $r = r(u^1, u^2)$ defines two-dimensional time-like surface F^2, the vectors ξ_1, ξ_2 are its space-like normal vectors, and g_{ij}, L_{ij}^k are the coefficients of the first and second quadratic forms, respectively. The number $H^k = g^{ij} L_{ij}^k$ is called the mean curvature of the surface for the direction of the normal vector ξ_k, and the vector $H = (H^1 \xi_1 + H^2 \xi_2)/2$ is the mean curvature vector. The time-like surfaces of Minkowski space with zero mean curvature vector will be called minimal surfaces, as in Euclidean space. We plan to apply the properties of the Grassman image of the minimal time-like surface to study its differential geometry, in particular, the question of the existence of such surfaces with some additional conditions on the Grassman image.

We can choose such a parameterization on the time-like surface F^2 in which $ds^2 = 2g_{12} du^1 du^2$. It follows from the minimal surface condition that $L_{12}^k = 0$. Then the system of Gauss-Codazzi-Ricci equations takes the form

$$\begin{cases} R_{1212} = L_{11}^1 L_{22}^1 + L_{11}^2 L_{22}^2, \\ (L_{11}^1)'_{u^2} = -\mu_{12/2} L_{11}^2, \\ (L_{11}^2)'_{u^2} = \mu_{12/2} L_{11}^1, \\ (L_{22}^1)'_{u^1} = -\mu_{12/1} L_{22}^2, \\ (L_{22}^2)'_{u^1} = \mu_{12/1} L_{22}^1, \\ (\mu_{12/1})'_{u^2} - (\mu_{12/2})'_{u^1} + (L_{11}^1 L_{22}^2 - L_{11}^2 L_{22}^1)\frac{1}{g_{12}} = 0, \end{cases} \tag{1}$$

where $\mu_{12/i}$ are torsion coefficients. These equations coincide with the equations in the work [1].

The Grassman image of two-dimensional surfaces is their important geometric characteristic. The work [2] shows that the non-degenerated Grassman image Γ^2 of the surface of Minkowski space is two-dimensional surface $p = p(u^1, u^2)$, which belongs to the four-dimensional Grassman submanifold $PG(2,4)$ of six-dimensional pseudo-Euclidean space ${}^3R_6$ of index 3. Tangent vectors to Γ^2 can be written in the form $p_{u_i} = -L_{ik}^1 g^{kl}[r_l, \xi_2] - L_{ik}^2 g^{kl}[\xi_1, r_l], l = 1, 2$.

This paper shows that the metric of the Grassman image of the minimal time-like surface of the space ${}^1R_4$ with respect to the basis $e_1 = \frac{r_1 - r_2}{\sqrt{2g_{12}}}$, $e_2 = \frac{r_1 + r_2}{\sqrt{2g_{12}}}$, $e_3 = \xi_1$, $e_4 = \xi_2$ has the form $ds^2 = \frac{L_{11}^1 L_{22}^1 + L_{11}^2 L_{22}^2}{g_{12}} du^1 du^2$, and therefore the Grassman image of the minimal time-like surface is also the time-like surface.

References

[1] Goroh V. P. O dvumernyh minimalnyh poverhnostyah psevdoevklidova prostranstva *Ukr. geometrich. sbornik*, 31: 36–47, 1988

[2] Grechneva, M. A., Stegantseva, P. G. On the existence of a surface in the pseudo-Euclidean space with given Grassman image. *Ukrainian Mathematical Journal*, 68 (10): 1320–1329, 2016