On the inverse Poletsky inequality with a cotangent dilatation

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The following definitions are from [1]. A path γ in $\mathbb{R}^n$ is a continuous mapping $\gamma:\Delta\rightarrow\mathbb{R}^n$ where Δ is an interval in $\mathbb{R}$. Its locus $\gamma(\Delta)$ is denoted by $|\gamma|$. Given a family Γ of paths γ in $\mathbb{R}^n$, a Borel function $\rho:\mathbb{R}^n\rightarrow[0,\infty]$ is called *admissible* for Γ, abbr. $\rho\in{\rm adm}\,\Gamma$, if

$$\int\limits_{\gamma}\rho(x)|dx|\geqslant 1$$

for each (locally rectifiable) $\gamma\in\Gamma$. Given $p\geqslant 1$, the *p-modulus* of Γ is defined by the relation

$$M_p(\Gamma):=\inf\limits_{\rho\in{\rm adm}\,\Gamma}\int\limits_{\mathbb{R}^n}\rho^p(x)dm(x) \qquad (1)$$

interpreted as $+\infty$ if ${\rm adm}\,\Gamma=\varnothing$.

We will need the following definitions related to paths, their lengths and mappings defined on them, see [2, section 8]. If $\gamma:\Delta\rightarrow\mathbb{R}^n$ is a locally rectifiable path, then there is the unique nondecreasing length function l_{γ} of Δ onto a length interval $\Delta_{\gamma}\subset\mathbb{R}$ with a prescribed normalization $l_{\gamma}(t_0)=0\in\Delta_{\gamma}$, $t_0\in\Delta$, such that $l_{\gamma}(t)$ is equal to the length of the subpath $\gamma|_{[t_0,t]}$ of γ if $t>t_0$, $t\in\Delta$, and $l_{\gamma}(t)$ is equal to minus length of $\gamma|_{[t,t_0]}$ if $t<t_0$, $t\in\Delta$. Let $g:|\gamma|\rightarrow\mathbb{R}^n$ be a continuous mapping, and suppose that the path $\widetilde{\gamma}=g\circ\gamma$ is also locally rectifiable. Then there is a unique non-decreasing function $L_{\gamma,g}:\Delta_{\gamma}\rightarrow\Delta_{\widetilde{\gamma}}$ such that $L_{\gamma,g}\left(l_{\gamma}(t)\right)=l_{\widetilde{\gamma}}(t)$ for all $t\in\Delta$. A path γ in D is called here a (whole) *lifting* of a path $\widetilde{\gamma}$ in $\mathbb{R}^n$ under $f:D\rightarrow\mathbb{R}^n$ if $\widetilde{\gamma}=f\circ\gamma$.

Further, we use the notation I for the segment $[a,b]$. Given a closed rectifiable path $\gamma:I\rightarrow\mathbb{R}^n$, we define a length function $l_{\gamma}(t)$ by the rule $l_{\gamma}(t)=S\left(\gamma,[a,t]\right)$, where $S(\gamma,[a,t])$ is the length of the path $\gamma|_{[a,t]}$. Let $\alpha:[a,b]\rightarrow\mathbb{R}^n$ be a rectifiable path in $\mathbb{R}^n$, $n\geqslant 2$, and $l(\alpha)$ be its length. A *normal representation* α^0 of α is defined as a path $\alpha^0:[0,l(\alpha)]\rightarrow\mathbb{R}^n$ which can be got from α by change of parameter such that $\alpha(t)=\alpha^0\left(S\left(\alpha,[a,t]\right)\right)$ for every $t\in[0,l(\alpha)]$. Such a normal representation always exists and is unique (see [**?**, Theorem 2.4]).

The following definition may be found in [1, 2.5, item 2, section I]. Let $\alpha:[a,b]\rightarrow\mathbb{R}^n$ be a closed rectifiable path in $\mathbb{R}^n$, $n\geqslant 2$. A mapping $f:|\alpha|\rightarrow\mathbb{R}^n$ is said to be *absolutely continuous on* α, if the function $f\circ\alpha^0$ is absolutely continuous on $[0,l(\alpha)]$, where $l(\alpha)$ denotes the length of α, and α^0 is its normal representation.

In the following, we say that some property P holds for *p-almost all paths in the domain* D if this property may be violated only for some family Γ_0 of paths in D such that $M_p(\Gamma_0)=0$, where $M_p(\Gamma_0)$ denotes the p-module of the family of paths Γ_0 defined in (1). We will say that the mapping $f:D\rightarrow\mathbb{R}^n$ has the *ACP-property with respect to p-modulus,* write $f\in ACP_p$, if the length function $L_{\gamma,f}$ is absolutely continuous on all closed intervals Δ_{γ} for p-almost all paths γ in D.

Let X and Y be two spaces with measures μ and μ', respectively. We say that a mapping $f:X\rightarrow Y$ has *N-property of Luzin*, if from the condition $\mu(E)=0$ it follows that $\mu'(f(E))=0$. Similarly, we say that a mapping $f : X \to Y$ has N^{-1}-*Luzin property,* if from the condition $\mu'(E) = 0$ it follows that $\mu(f^{-1}(E)) = 0$.

Let $x \in D$ be a differentiability point of f. We set

$$l(f'(x)) = \min_{h\in\mathbb{R}^n\setminus\{0\}} \frac{|f'(x)h|}{|h|}, \quad \|f'(x)\| = \max_{h\in\mathbb{R}^n\setminus\{0\}} \frac{|f'(x)h|}{|h|}, \quad J(x,f) = \det f'(x)\,.$$

Given sets E and F and a given domain D in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, we denote by $\Gamma(E,F,D)$ the family of all paths $\gamma : [0,1] \to \overline{\mathbb{R}^n}$ joining E and F in D, that is, $\gamma(0) \in E$, $\gamma(1) \in F$ and $\gamma(t) \in D$ for all $t \in (0,1)$. Everywhere below, unless otherwise stated, the boundary and the closure of a set are understood in the sense of the extended Euclidean space $\overline{\mathbb{R}^n}$. Let $x_0 \in \overline{D}$, $x_0 \neq \infty$,

$$B(x_0,r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}\,, \quad \mathbb{B}^n = B(0,1)\,,$$
$$S(x_0,r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}\,, S_i = S(x_0,r_i)\,, \quad i = 1,2\,,$$
$$A = A(x_0,r_1,r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}\,.$$

Let $f : D \to \mathbb{R}^n$, $n \geqslant 2$, and let $Q : \mathbb{R}^n \to [0,\infty]$ be a Lebesgue measurable function such that $Q(y) \equiv 0$ for $y \in \mathbb{R}^n \setminus f(D)$. Let $A = A(y_0,r_1,r_2)$ and $\Gamma_f(y_0,r_1,r_2)$ denotes the family of all paths $\gamma : [a,b] \to D$ such that $f(\gamma) \in \Gamma(S(y_0,r_1), S(y_0,r_2), A(y_0,r_1,r_2))$, i.e., $f(\gamma(a)) \in S(y_0,r_1)$, $f(\gamma(b)) \in S(y_0,r_2)$, and $f(\gamma(t)) \in A(y_0,r_1,r_2)$ for any $a < t < b$. We say that f *satisfies the inverse Poletsky inequality at* $y_0 \in f(D)$ *with respect to p-modulus,* if the relation

$$M_p(\Gamma_f(y_0,r_1,r_2)) \leqslant \int\limits_A Q(y)\cdot \eta^p(|y - y_0|)\,dm(y) \tag{2}$$

holds for any $0 < r_1 < r_2 < r_0 := \sup\limits_{y\in f(D)} |y - y_0|$ and any Lebesgue measurable function $\eta : (r_1,r_2) \to [0,\infty]$ such that $\int\limits_{r_1}^{r_2} \eta(r)\,dr \geqslant 1$. A mapping $f : D \to \mathbb{R}^n$ is called *weakly light,* if, for any $y \in \mathbb{R}^n$, each connected component $\{f^{-1}(y)\}$ does not contain a non-degenerate path (see, e.g., Remark 8.3 in [2]).

Theorem 1. *Let $p > 1$, and let $f : D \to \mathbb{R}^n$ be a weakly light mapping which is differentiable a.e. and has Luzin N- and N^{-1}-properties with respect to the Lebesgue measure in $\mathbb{R}^n$, besides that, $f \in ACP_p(D)$. Let $y_0 \in \overline{f(D)} \setminus \{\infty\}$. Set*

$$K_{CT,p,y_0}(y,f) = \sum_{x\in f^{-1}(y)} \frac{\left(\sup\limits_{|h|=1}\left|\left(f'(x)h, \frac{f(x)-y_0}{|f(x)-y_0|}\right)\right|\right)^p}{|J(x,f)|}\,. \tag{3}$$

Then f satisfies the inverse Poletsky inequality 2 at y_0 for $Q_*(y) := K_{CT,p,y_0}(y,f)$.

The result mentioned above is published in [3].

References

[1] Väisälä J. *Lectures on n-Dimensional Quasiconformal Mappings.* Lecture Notes in Math. 229. Berlin etc., Springer–Verlag, 1971.

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