Hasse norm theorem for 3-manifolds

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Abstract:Following the analogies between knots and primes, 3-manifolds and number rings in arithmetic topology, we show a topological analogue of the Hasse norm principle for finite cyclic coverings of 3-manifolds, which was originally stated for finite cyclic extensions of number fields.

Theorem 1. *Let M be an integral homology 3-sphere endowed with a very admissible link $\mathcal{L}$. Let $f : N \to M$ be a finite cyclic covering branched over a finite sublink L_0 of $\mathcal{L}$. Then,*

$$P_{M,\mathcal{L}} \cap f_*(I_{N,f^{-1}(\mathcal{L})})) = f_*(P_{N,f^{-1}(\mathcal{L})}).$$

Lemma 2. *Let M be an oriented connected closed 3-manifold endowed with a very admissible link $\mathcal{L}$. Let $f : N \to M$ be a finite covering branched over a finite link $L_0 \subset \mathcal{L}$. Let $f_* : I_{N,f^{-1}(\mathcal{L})} \to I_{M,\mathcal{L}}$ denote the homomorphism induced by f. Then, we have*

$$f_*(\prod_{J \subset f^{-1}(\mathcal{L})} \mathbb{Z}[\mu_J]) \subset \prod_{K \subset \mathcal{L}} \mathbb{Z}[\mu_K].$$

Proposition 3. *Let M be an integer homology 3-shpere endowed with a very admissible link $\mathcal{L}$ and $[A] \in H_2(M, \mathcal{L})$. Then there is a finite sublink $L \subset \mathcal{L}$ such that $[A] \in H_2(M, L)$. We can write $[A] = \sum_{K \subset L} c_K[S_K]$ with $c_K \in \mathbb{Z}$. Let $\Delta_{M,\mathcal{L}}([A]) = (a_K)_{K \subset \mathcal{L}} \in I_{M,\mathcal{L}}$. Then we have the following formula:*

$$a_K = \begin{cases} c_K[\lambda_K] - (\sum_{K' \subset L \setminus K} \mathrm{lk}(K, K')c_{K'})[\mu_K] & (K \subset L) \\ -\sum_{K' \subset L} \mathrm{lk}(K, K')c_{K'}[\mu_K] & (K \subset \mathcal{L} \setminus L) \end{cases}$$

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