Geometric and algebraic properties of dispersionless Nizhnik equation

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The dispersionless Nizhnik equation (see [1] for justifying this name)

$$u_{txy} = (u_{xx}u_{xy})_x + (u_{xy}u_{yy})_y \tag{1}$$

is the dispersionless limit of the symmetric Nizhnik equation, which is the potential equation of the Nizhnik system [3] in the symmetric case. The equation (1) has interesting geometric and algebraic properties. In particular, the maximal Lie invariance (pseudo)algebra $\mathfrak{g}$ of (1) is infinite-dimensional and is spanned by the vector fields

$$\begin{gathered}D^t(\tau) = \tau\partial_t + \tfrac{1}{3}\tau_t x\partial_x + \tfrac{1}{3}\tau_t y\partial_y - \tfrac{1}{18}\tau_{tt}(x^3+y^3)\partial_u, \quad D^{\rm s} = x\partial_x + y\partial_y + 3u\partial_u,\\ P^x(\chi) = \chi\partial_x - \tfrac{1}{2}\chi_t x^2\partial_u, \quad P^y(\rho) = \rho\partial_y - \tfrac{1}{2}\rho_t y^2\partial_u,\\ R^x(\alpha) = \alpha x\partial_u, \quad R^y(\beta) = \beta y\partial_u, \quad Z(\sigma) = \sigma\partial_u,\end{gathered}$$

where τ, χ, ρ, α, β and σ run through the set of smooth functions of t. Moreover, the contact invariance (pseudo)algebra $\mathfrak{g}_{\rm c}$ of (1) coincides with the first prolongation of the algebra $\mathfrak{g}$.

The point- and contact-symmetry pseudogroups G and $G_{\rm c}$ of (1) were efficiently constructed in [1] by using the original version of the algebraic megaideal-based method suggested in [2]. The basic (necessary) method condition that the pushforward Φ_* of elements $\mathfrak{g}$ by any element Φ of G preserves any megaideal $\mathfrak{m}$ of $\mathfrak{g}$, $\Phi_*\mathfrak{m} \subseteq \mathfrak{m}$, is replaced in this version by a weaker but more computationally efficient condition $\Phi_*(\mathfrak{m}\cap\mathfrak{s}) \subseteq \mathfrak{m}$ for an arbitrary essential megaideal $\mathfrak{m}$ and a selected fixed finite-dimensional subalgebra $\mathfrak{s}$ of $\mathfrak{g}$. As such $\mathfrak{s}$ for (1), we can take $\mathfrak{s}_1 = \mathfrak{a} \ni \langle D^{\rm s}\rangle$ or $\mathfrak{s}_2 = \mathfrak{a} \ni \langle D^t(1), D^t(t)\rangle$, where $\mathfrak{a} = \big\langle Z(1), Z(t), R^z(1), P^z(1), P^z(t), z\in\{x,y\}\big\rangle$.

Theorem 1. *(i) The point-symmetry pseudogroup G of the dispersionless Nizhnik equation* (1) *is generated by the transformations of the form*

$$\begin{gathered}\tilde t = T(t), \quad \tilde x = CT_t^{1/3}x + X^0(t), \quad \tilde y = CT_t^{1/3}y + Y^0(t),\\ \tilde u = C^3u - \frac{C^3T_{tt}}{18T_t}(x^3+y^3) - \frac{C^2}{2T_t^{1/3}}(X^0_tx^2 + Y^0_ty^2) + W^1(t)x + W^2(t)y + W^0(t)\end{gathered}$$

and the transformation $\mathscr{J}$: $\tilde t = t$, $\tilde x = y$, $\tilde y = x$, $\tilde u = u$. Here T, X^0, Y^0, W^0, W^1 and W^2 are arbitrary smooth functions of t with $T_t \neq 0$, and C is an arbitrary nonzero constant.

(ii) The contact-symmetry pseudogroup $G_{\rm c}$ of the dispersionless Nizhnik equation (1) *coincides with the first prolongation $G_{(1)}$ of the pseudogroup G.*

Thus, a complete list of independent discrete point symmetry transformations of (1) is exhausted by three commuting involutions, $\mathscr{J}$, $\mathscr{I}$ and $\mathscr{I}$, which map (t,x,y,u) to (t,y,x,u), $(-t,-x,-y,u)$ and $(t,-x,-y,-u)$, respectively.

The equation (1) is peculiar due to the fact that the condition $\Phi_*\mathfrak g\subseteq\mathfrak g$ completely defines G and thus is not only necessary but also sufficient in this particular case. The similar claim holds for $\mathfrak g_{\rm c}$ and $G_{\rm c}$. This is the first and so far the only example of this kind in the literature.

In the context of the method applied, an important problem is to select certain subalgebras of $\mathfrak g$ and $\mathfrak g_{\rm c}$.

Definition 2. We call a proper subalgebra $\mathfrak s$ of a Lie algebra $\mathfrak a$ of vector fields a *subalgebra defining the diffeomorphisms that stabilize* $\mathfrak a$ if the conditions $\Phi_*\mathfrak a\subseteq\mathfrak a$ and $\Phi_*\mathfrak s\subseteq\mathfrak a$ for local diffeomorphisms Φ in the underlying space are equivalent.

Theorem 3. *The subalgebra $\mathfrak s_2$ of the algebra $\mathfrak g$ defines the diffeomorphisms that stabilize $\mathfrak g$, whereas the subalgebra $\mathfrak s_1$ and even the subalgebra $\bar{\mathfrak s}_1:=\mathfrak s_1+\langle D^t(1)\rangle$ does not have this property.*

Corollary 4. *The first prolongation of $\mathfrak s_2+\mathfrak s_3$ with $\mathfrak s_3:=\langle Z(1),Z(t),Z(t^2),R^z(1),R^z(t),z\in\{x,y\}\rangle$, which is a subalgebra of $\mathfrak g_{\rm c}=\mathfrak g_{(1)}$, defines the diffeomorphisms of the corresponding first-order jet space that stabilize $\mathfrak g_{\rm c}$.*

We also found geometric properties of the dispersionless Nizhnik equation (1) that completely define this equation. Although the maximal Lie invariance algebra $\mathfrak g$ of the equation (1) exhaustively defines the point-symmetry pseudogroup G of this equation, it does not define exhaustively the equation itself.

Lemma 5. *(i) A partial differential equation of order less than or equal to three with three independent variables is invariant with respect to the algebra $\mathfrak g$ if and only if it is of the form*

$$u_{txy}=(u_{xx}u_{xy})_x+(u_{xy}u_{yy})_y+u_{xy}u_{xyy}H(\omega_1,\omega_2),\quad \omega_1:=\frac{u_{xxx}-u_{yyy}}{u_{xyy}},\quad \omega_2:=\frac{u_{xxy}}{u_{xyy}},\tag{2}$$

where H is an arbitrary smooth function of its arguments.

(ii) An equation of the form (2) admits the conservation-law characteristic 1 and thus it is in conserved form if and only if H is an affine function of (ω_1,ω_2), i.e., $H=a\omega_1+b\omega_2+c$ for some constants a, b and c, and the equation takes the form

$$u_{txy}=(u_{xx}u_{xy})_x+(u_{xy}u_{yy})_y+u_{xy}\big(a(u_{xxx}-u_{yyy})+bu_{xxy}+cu_{xyy}\big).\tag{3}$$

(iii) An equation of the form (3) admits the conservation-law characteristic u_{xx} or u_{yy} if and only if $a=b=0$ or $a=c=0$, respectively.

Theorem 6. *An rth order ($r\in\{1,2,3\}$) partial differential equation with three independent variables admits the algebra $\mathfrak g$ as its Lie invariance algebra and the conservation-law characteristics 1, u_{xx} and u_{yy} if and only if it coincides with the dispersionless Nizhnik equation (1).*

The presented properties of the equation (1) are used in [4] to construct its exact solutions.

References

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