Topology of the Hilbert Schemes of monomial plane curve singularities

Masahiro Watari
(University of Kuala Lumpur, Malaysia France Institute, Japanese Collaboration Program
Section 14, Jalan Damai, Seksyen 14, 43650
Bandar Baru Bangi, Selengor, Malaysia)
E-mail: masahiro@unikl.edu.my

Let X be a singular irreducible plane curve over $\mathbb{C}$. For a singular point o of X, we refer to the pair (X, o) as a plane curve singularity. Let $\mathcal{O}_{X,o}$ (resp. Γ) be the local ting of (X, o) (resp. the semi-group associated with (X, o)). We denote by $\mathrm{Hilb}^r(X, o)$ the punctual Hilbert scheme of degree r for a given singularity (X, o). Piontkowski [1] studied the topology of the Jacobian factor $J_{X,o}$ for a plane curve singularity (X, o) with $\Gamma = \langle p, q\rangle$ $(\gcd(p, q) = 1)$. He showed the existence of an affine cell decomposition of the Jocobi factor $J_{X,o}$ and the Euler number of J_X and the Betti numbers of J_X are described. In this talk, we generalize Piontkowski's results to the cases of the punctual Hilbert schemes of (X, o).

In this talk, we always consider the plane curve singularity whose local ring $\mathcal{O}_{X,o}$ is $\mathbb{C}[[t^p, t^q]]$ where $\gcd(p, q) = 1$. Then such a singularity has $\Gamma = \langle p, q\rangle$ as its semi-group. Let $\mathrm{Mod}(\Gamma)$ be the set of all Γ-semi-modules. Defining $\mathrm{codim}\Delta := \#(S \setminus \Delta)$, we set $\mathrm{Mod}^r(\Gamma) := \{\Delta \in \mathrm{Mod}(\Gamma) |\, \mathrm{codim}\Delta = r\}$. It is known that the components of $\mathrm{Hilb}^r(X, o)$ is parametrized by the elements of $\mathrm{Mod}^r(\Gamma)$.

$$\mathrm{Hilb}^r(X, o) = \bigcup_{\Delta\in\mathrm{Mod}^r(\Gamma)} H(\Delta) \tag{1}$$

The component $H(\Delta)$ in (1) is called the Δ-subset of $\mathrm{Hilb}^r(X, o)$.

Theorem 1. *Let (X, o) be a plane curve singularity whose local ring $\mathcal{O}_{X,o}$ is $\mathbb{C}[[t^p, t^q]]$ where $\gcd(p, q) = 1$. Each Δ-subset $H(\Delta)$ in (1) is isomorphic to an affine space whose dimension is given by*

$$\sum_{i=1}^{p-1} \#[\{(\Gamma - \min \Delta) \cap [a_i, a_i + q]\} \setminus \Delta^{(0)}]. \tag{2}$$

Here $\Delta^{(0)}$ is the 0-nomalization of Δ and $\{a_0, \dots, a_{p-1}\}$ is the p-basis of $\Delta^{(0)}$.

The following fact follows from Theorem 1

Corollary 2. *Let (X, o) be a plane curve singularity with $\mathcal{O}_{X,o} = \mathbb{C}[[t^p, t^q]]$ $(\gcd(p, q) = 1)$. The Euler number of $\mathrm{Hilb}^r(X, o)$ is equal to $\#\mathrm{Mod}^r(\Gamma)$.*

We denote by $e(\mathrm{Hilb}^r(X, o))$ the Euler number of $\mathrm{Hilb}^r(X, o)$.

Example 3. The Euler numbers of the punctual Hilbert schemes for the A_{2l}-singularity are given in the following table:

r	$0 \le r \le 2l$	$r \ge 2l + 1$
$e(\mathrm{Hilb}^r(X, o))$	$[r/2] + 1$	$l + 1$

Here the notaion $[a]$ $(a \in \mathbb{R})$ is the biggest integer that is smaller than a.

Setting $\mathrm{codim}\, H(\Delta) := \dim \mathrm{Hilb}^r(X, o) - \dim\, H(\Delta)$, we define

$$\mathcal{H}_{r,d} := \{H(\Delta) |\, \Delta \in \mathrm{Mod}^r(\Gamma) \text{ and } \dim H(\Delta) = d\},$$
$$\mathcal{H}_r^d := \{H(\Delta) |\, \Delta \in \mathrm{Mod}^r(\Gamma) \text{ and codim}\, H(\Delta) = d\}.$$

Theorem 4. *Let (X, o) be a plane curve singularity with the local ring* $\mathbb{C}[[t^p, t^q]]$ $(\gcd(p,q) = 1)$. *Then the odd (co-) homology groups of* $\mathrm{Hilb}^r(X, o)$ *are zero. The even (co-) homology groups of* $\mathrm{Hilb}^r(X, o)$ *are free abelian groups with Betti numbers*

$$h_{2d}(\mathrm{Hilb}^r(X, o)) = \#\mathcal{H}_{r,d} \text{ and } h^{2d}(\mathrm{Hilb}^r(X, o)) = \#\mathcal{H}_r^d.$$

Example 5. The even (co-) homology groups of $\mathrm{Hilb}^r(X, o)$ for the A_{2l}-singularity are given in the following table:

r	$0 \le r \le 2l$	$r \ge 2l+1$
d	$0 \le d \le r$	$0 \le d \le l$
$h_{2d}(\mathrm{Hilb}^r(X, o))$	1	1
$h^{2d}(\mathrm{Hilb}^r(X, o))$	1	1

References

[1] J. Piontkowski, Topology of the compactified Jacobians of singular curves. Math. Z. **255** (2007), 195–226.