Self-similar actions of the fundamental group of the Klein bottle

Davyd Zashkolnyi
(Taras Shevchenko National University of Kyiv, Kyiv 83000)
E-mail: davendiy@gmail.com

A virtual endomorphism of a group G is a homomorphism of the form $\phi : H \to G$, where $H < G$ is a subgroup of finite index. A virtual endomorphism $\phi : H \to G$ is called simple if there are no nontrivial normal ϕ-invariant subgroups.

A recursive construction using a simple virtual endomorphism ϕ produces a so-called self-similar action of the group G on a d-regular rooted tree X^*. The X^* represents words over the alphabet X of size d. In general, a faithful action of a group G on rooted tree X^* is said to be *self-similar* if for every $g \in G$ and every $x \in X$ there exists unique pair $g|_x \in G$ and $y \in X$ such that $g(xw) = yg|_x(w)$. A self-similar action is called *self-replicating* if the associative simple virtual endomorphism ϕ is surjective. One can find more information regarding self-similar actions in [1].

Consider the fundamental group of the Klein bottle K. The group K is finitely generated by affine transformations $t(x, y) = (x, y + 1)$ and $s(x, y) = (x + 1/2, -y)$. We can show that for every virtual endomorphism $\phi : H \to K$ there exist subgroup of finite index $H_1 \sim \mathbb{Z}^2$ and associated matrix $B_\phi \in M_2(\mathbb{Q})$ of rational entities such that the restriction $\phi|_{H_1} : H_1 \to K$ is in fact a linear map $\phi|_{H_1}(x) = B_\phi x$.

Theorem 1. *Let $\phi : H \to K$ be a virtual endomorphism and $B_\phi \in M_2(\mathbb{Q})$ the associated matrix. Then ϕ is simple, and therefore produces a self-similar action, if and only if B_ϕ is not of the forms:*

$$\begin{pmatrix} \alpha & \frac{n}{m}\beta \\ \frac{m}{n}\gamma & \delta \end{pmatrix}, \frac{1}{2}\begin{pmatrix} \alpha+\beta+\gamma+\delta & \frac{n}{m}(\alpha+\beta-\gamma-\delta) \\ \frac{m}{n}(\alpha-\beta+\gamma-\delta) & \alpha-\beta-\gamma+\delta \end{pmatrix}, \begin{pmatrix} k & b_1 \\ 0 & b_2 \end{pmatrix} \text{ or } \begin{pmatrix} b_1 & 0 \\ b_2 & k \end{pmatrix} \quad (1)$$

for $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$; $n, m, k \in \mathbb{Z}$; $b_i \in \mathbb{Q}$.

Theorem 2. *1) The group K admits a transitive self-similar action on a d-regular rooted tree if and only if $d \geq 2$ is not an odd prime.*

2) The group K admits a self-replicating action on a d-regular rooted tree if and only if d is not a prime or a power of 2.

A self-similar action (G, X^*) is called *finite-state* if for every $g \in G$ the set of its sections $\{g|_v : v \in X^*\}$ is finite. In other words, the action of every element can be emulated by a finite-state transducer.

A self-similar action (G, X^*) is called *contracting* if there exists finite set $\mathcal{N} \subset G$ such that for every $g \in G$ there exists $n \in \mathbb{N}$ such that $g|_v \in \mathcal{N}$ for all $v \in X^*$ of length $\geq n$.

Theorem 3. *Let (K, X^*) be a transitive self-similar action and B_ϕ the matrix of the associated virtual endomorphism ϕ.*

1) The (K, X^*) is contracting if and only if the eigenvalues of B_ϕ are less than 1 in modulus.

2) If (K, X^*) is self-replicating, then (K, X^*) is finite-state if and only if it is contracting.

3) The group K admits a transitive finite-state (contracting) action of degree d if and only if $d \geq 2$ is not an odd prime.

References

[1] Nekrashevych, V.: *Self-similar groups.* Providence, RI: American Mathematical Society, 2005.