Balayage on locally compact spaces

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This talk is based on [1]–[3], and it is devoted to the theory of potentials on a locally compact (Hausdorff) space X with respect to a *kernel* κ, κ being thought of as a symmetric, lower semicontinuous function $\kappa : X \times X \to [0, \infty]$. To be exact, we are interested in generalizations of the classical theory of balayage (sweeping out) on $\mathbb{R}^n$, $n \geqslant 2$ (see e.g. [4]–[7]), to a suitable kernel κ on X.

We denote by $\mathfrak{M}$ the linear space of all (real-valued Radon) measures μ on X, equipped with the *vague* topology of pointwise convergence on the continuous functions $\varphi : X \to \mathbb{R}$ of compact support, and by $\mathfrak{M}^+$ the cone of all positive $\mu \in \mathfrak{M}$. (For the theory of measures and integration on X, we refer to Bourbaki [8].) Given $\mu, \nu \in \mathfrak{M}$, the *mutual energy* and the *potential* are introduced by

$$I(\mu, \nu) := \int \kappa(x, y)\, d(\mu \otimes \nu)(x, y) \quad \text{and} \quad U^\mu(x) := \int \kappa(x, y)\, d\mu(y), \quad x \in X,$$

respectively, provided the value on the right is well defined as a finite number or $\pm\infty$. For $\mu = \nu$, the mutual energy $I(\mu, \nu)$ defines the *energy* $I(\mu, \mu) =: I(\mu)$ of $\mu \in \mathfrak{M}$.

In what follows, a kernel κ is assumed to satisfy the *energy principle*, which means that $I(\mu) \geqslant 0$ for all (signed) $\mu \in \mathfrak{M}$, and moreover that $I(\mu) = 0$ only for $\mu = 0$. Then all $\mu \in \mathfrak{M}$ of finite energy form a pre-Hilbert space $\mathcal{E}$ with the inner product $\langle \mu, \nu \rangle := I(\mu, \nu)$ and the energy norm $\|\mu\| := \sqrt{I(\mu)}$, cf. [9, Lemma 3.1.2]. The topology on $\mathcal{E}$ introduced by means of this norm is said to be *strong*.

In addition, we shall always assume that κ satisfies the *consistency* principle, which means that the cone $\mathcal{E}^+ := \mathcal{E} \cap \mathfrak{M}^+$ is *complete* in the induced strong topology, and that the strong topology on $\mathcal{E}^+$ is *finer* than the vague topology on $\mathcal{E}^+$; such a kernel is said to be *perfect* (Fuglede [9]). Thus any strong Cauchy net $(\mu_j) \subset \mathcal{E}^+$ converges *both strongly and vaguely* to the same unique measure $\mu_0 \in \mathcal{E}^+$.

Yet another permanent requirement on κ is that it satisfies the *domination principle*, which means that for any $\mu \in \mathcal{E}^+$ and any $\nu \in \mathfrak{M}^+$ with $U^\mu \leqslant U^\nu$ μ-a.e., the same inequality holds on all of X.

For any $A \subset X$, we denote by $\mathfrak{C}_A$ the upward directed set of all compact subsets K of A, where $K_1 \leqslant K_2$ if and only if $K_1 \subset K_2$. If a net $(x_K)_{K \in \mathfrak{C}_A} \subset Y$ converges to $x_0 \in Y$, Y being a topological space, then we shall indicate this fact by writing: $x_K \to x_0$ in Y as $K \uparrow A$.

Given $A \subset X$, we denote by $\mathfrak{M}^+_A$ the class of all $\mu \in \mathfrak{M}^+$ *concentrated on* A, which means that $X \setminus A$ is locally μ-negligible, or equivalently that A is μ-measurable and $\mu = \mu|_A$, $\mu|_A$ being the trace of μ to A. Also write $\mathcal{E}^+_A := \mathfrak{M}^+_A \cap \mathcal{E}$, and let $\mathcal{E}'_A$ stand for the closure of $\mathcal{E}^+_A$ in the strong topology on $\mathcal{E}$. Being a strongly closed subcone of the strongly complete cone $\mathcal{E}^+$, $\mathcal{E}'_A$ is likewise strongly complete.

Denoting by $c_*(E)$ and $c^*(E)$ the *inner* and *outer* capacity of $E \subset X$, respectively [9, Section 2.3], we say that an assertion $\mathcal{A}(x)$ involving a variable point $x \in X$, holds *nearly everywhere* (*n.e.*), resp. *quasi-everywhere* (*q.e.*), on a set A if $c_*(E) = 0$, resp. $c^*(E) = 0$, where $E := \{x \in A : \ \mathcal{A}(x) \text{ fails}\}$.

Theorem 1. *For any $A \subset X$ and any $\sigma \in \mathcal{E}^+$, there exists $\sigma^A \in \mathcal{E}'_A$, called the inner balayage of σ to A, that is uniquely characterized by any one of the following (equivalent) assertions.*

(i) *σ^A is the unique measure in $\mathcal{E}'_A$ having the property*

$$U^{\sigma^A} = U^\sigma \quad \text{n.e. on } A.$$

(ii) *σ^A is the unique orthogonal projection of σ in the pre-Hilbert space $\mathcal{E}$ onto the (convex, strongly complete) cone $\mathcal{E}'_A$, that is, $\sigma^A \in \mathcal{E}'_A$ and*

$$\|\sigma - \sigma^A\| = \min_{\mu \in \mathcal{E}'_A} \|\sigma - \mu\|.$$

(iii) σ^A *is the unique measure in* $\mathcal{E}^+$ *satisfying any one of the following three limit relations:*

$$\sigma^K \to \sigma^A \quad \text{strongly in } \mathcal{E}^+ \text{ as } K \uparrow A,$$
$$\sigma^K \to \sigma^A \quad \text{vaguely in } \mathcal{E}^+ \text{ as } K \uparrow A,$$
$$U^{\sigma^K} \uparrow U^{\sigma^A} \quad \text{pointwise on } X \text{ as } K \uparrow A,$$

where σ^K *denotes the only measure in* $\mathcal{E}^+_K$ *with the property* $U^{\sigma^K} = U^\sigma$ *n.e. on* K.

(iv) σ^A *is the only measure in the class* $\Gamma_{A,\sigma}$ *having the property*

$$U^{\sigma^A} = \min_{\mu\in\Gamma_{A,\sigma}} U^\mu \quad \text{on all of } X, \tag{1}$$

where $\Gamma_{A,\sigma} := \bigl\{\mu \in \mathcal{E}^+ : \ U^\mu \geqslant U^\sigma \quad \text{n.e. on } A\bigr\}$.

(v) σ^A *is the only measure in the class* $\Gamma_{A,\sigma}$*, introduced above, with the property*

$$\|\sigma^A\| = \min_{\mu\in\Gamma_{A,\sigma}} \|\mu\|.$$

Theorem 2. *If a space* X *is second-countable, while a set* A *is Borel, then Theorem* 1 *remains valid with "n.e. on* A*" replaced throughout by "q.e. on* A*". The measure* ω^{*A}*, thereby uniquely determined, is said to be the outer balayage of* ω *onto* A*. Actually,* $\omega^{*A} = \omega^A$*.* (Compare with [10, Theorem 4.12].)

Remark 3. All the above-mentioned assumptions on a space X and a kernel κ are fulfilled by:

- ✓ The α-Riesz kernels $|x-y|^{\alpha-n}$ of order $\alpha \in (0,2]$, $\alpha < n$, on $\mathbb{R}^n$, $n \geqslant 2$ (see [6, Theorems 1.15, 1.18, 1.27, 1.29]).
- ✓ The associated α-Green kernels, where $\alpha \in (0,2]$ and $\alpha < n$, on an arbitrary open subset of $\mathbb{R}^n$, $n \geqslant 2$ (see [11, Theorems 4.6, 4.9, 4.11]).
- ✓ The (2-)Green kernel on a planar Greenian set (see [5] and [7, Sections I.V.10, I.XIII.7]).

(We emphasize that some of the results formulated above are new even for these classical kernels.)

Remark 4. The theory of balayage thereby developed has already been shown to be a powerful tool in the well-known Gauss variational problem, see [12].

Problem 5. What kind of additional assumptions on X and κ would make it possible to generalize the theory of balayage, presented above, to Radon measures on X of *infinite* energy?

References

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