Multiplicative b-homogeneralized Derivations of Associative Rings

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Abstract

In this manuscript, we present multiplicative b-homogeneralized derivation on an associative ring R and discuss certain differential (functional) identities having multiplicative b-homogeneralized derivation. Investigating the centralizer of suitable subset over semiprime rings that admit multiplicative b-homogeneralized derivation enhances some outcomes in the literature. We refer the reader to [4] and [2] for more details.

Introduction

♣ As is well known, the problem of linear mappings preserving fixed products is a very interesting item in the field of operator algebra. Derivations that can be completely determined by the local action on some subsets of algebra have attracted attention of many researchers. The Martindale ring of quotients of a prime ring R was introduced in W. S. Martindale [6] as a tool for studying rings satisfying a polynomial identity. The concept was extended to semiprime rings in S. A. Amitsur [5]. Historically, the study of derivation was initiated during the 1950s and 1960s. Derivations of rings got a tremendous development in 1957, when E.C. Posner [3] established two very striking results in the case of prime rings.

♣ When is a multiplicative derivation additive? This question was issued via Martindale. Even in 1991, an affirmative answer to this question was created by Daif [8], where he introduced the idea of multiplicative derivation and provided: Further, in continuation of this study, Daif and Tammam El-Sayiad [9] presented the idea of multiplicative generalized derivation. They discussed a similar situation for the additivity of multiplicative generalized derivation.

♣ Recently, Mehsin [7] investigated the behavior of certain differential identities involving three multiplicative generalized (λ, λ)-derivations on the ideals of semiprime rings. Where he proved that a ring R has commutative ideals with the help of $(F, f), (G, g)$ and (H, h) are multiplicative generalized (λ, λ)-derivations on ideals in semiprime rings with some conditions.

♣ Named that an associative R is a semiprime when R satisfy the expression $r_1 R r_1 = 0$ which yields $r_1 = 0$ and R is prime if $r_1 R r_2 = 0$ which supply two options there either $r_1 = 0$ or $r_2 = 0$. As a factual information about the connection between the previous concepts a prime and semiprime ring mentioned as following: A prime ring forms another kind of ring, which is a semiprime, while the converse, unfortunately, is not always true. Let Q_{mr} and Q_s denote its right Utumi quotient ring and right symmetric Martindale quotient ring, respectively.

A left ideal K of a ring R is said to be dense if for every $x, y \in R$, with $x \neq 0$, there exists $a \in R$ such that $ax \neq 0$ and $ay \in K$.

♣ When a ring R admits for all $r_1, r_2 \in R$ satisfying Leibniz's rule, which is $d(r_1 r_2) = d(r_1)r_2 + r_1 d(r_2)$ then a derivation is that an additive map $d\colon R \to R$. Whenever for all $r_1, r_2 \in R$ there exists an identity $D(r_1 r_2) = D(r_1)r_2 + r_1 d(r_2)$. Then, D is an additive mapping defined as $D\colon R \to R$ is recorded as a generalized, *i.e.* a generalized derivation, where d worked as an additive mapping derivation over R.

In 2000, a classical definition of homoderivation posted via El Sofy's article [1], where he was described an additive mapping a homoderivation concerning a ring R like ψ from R to R satisfying $\psi(xy) = \psi(x)\psi(y) + \psi(x)y + x\psi(y)$ where x and y belong to R. An example of such mapping is to let $\psi(x) = \psi(x) - x$ for all $x \in R$ where ψ is an endomorphism on R. It is clear that a homoderivation ψ is also a derivation if $\psi(x)\psi(y) = 0$ for all $x, y \in R$.

An additive mapping $F\colon R \to R$ is called a right generalized homoderivation if there exists a homoderivation $\psi\colon R \to R$ such that $F(xy) = F(x)\psi(y) + F(x)y + x\psi(y)$ for all $x, y \in R$.

♣ Moreover, mapping $F\colon R \to Q_{mr}$ associated with derivation (need not be additive) $d\colon R \to R$ such that $F(\sigma\tau) = F(\sigma)\tau + b\sigma d(\tau)$ holds for all $\sigma, \tau \in R$ and any fixed $0 \neq b \in Q_s \subset Q_{mr}$. If F is additive (not necessarily additive), then F is called b-generalized derivation (multiplicative) b-generalized.

♣ Hence, one may observe that the concept of multiplicative generalized derivations includes the concept of derivations, generalized derivations and the left multipliers (*i.e.*, $F(xy) = F(x)y$ for all $x, y \in R$). So, it should be interesting to extend some results concerning these notions to multiplicative generalized derivations. Every generalized derivation is a multiplicative generalized derivation. But the converse is not true in general.

In the next slide, we will see the main definition, which represents the generalization of the previous definition.

The Main Definition

♣ Suppose that R is an associative ring, mapping $F\colon R \to Q_{mr}$ associated with homoderivation $d\colon R \to R$ such that $F(\sigma\tau) = F(\sigma)d(\tau) + F(\sigma)\tau + b\sigma d(\tau)$ holds for all $\sigma, \tau \in R$ and any fixed $0 \neq b \in Q_s \subset Q_{mr}$. When F (is not necessarily additive), then F is called ***b*-homogeneralized derivation (multiplicative *b*-homogeneralized)**.

To be more close to the previous definition, we push the following example:

♣ Let $R = \{ \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & \lambda \\ 0 & 0 & 0 \end{pmatrix} \setminus \text{for all } \alpha, \beta, \lambda \in \mathbb{F}\}$ be a ring of matrices over a field $\mathbb{F}$. Let d and F be the two additive mappings of R, given by:

$F(x) = \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & \lambda \\ 0 & 0 & 0 \end{pmatrix}$ and $d(x) = \begin{pmatrix} 0 & 0 & \alpha^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where $x \in R$ such that

$x = \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & \lambda \\ 0 & 0 & 0 \end{pmatrix}$. First of all, we should see that d acts as a b-homoderivation of R. *i.e.* $d(xy) = d(x)d(y) + d(x)y + bxd(y)$ for all $x, y \in R$. The left side produces

$d(xy) = \begin{pmatrix} 0 & 0 & \alpha\sigma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which equal to the right side, where

$y \in R$ such that $y = \begin{pmatrix} 0 & \tau & \nu \\ 0 & 0 & \sigma \\ 0 & 0 & 0 \end{pmatrix}$.

Hence, d is a b-homoderivation of R. Moreover, we investigate the formula $F(xy) = F(x)d(y) + F(x)y + bxd(y)$ for all $x, y \in R$. Here

$F(xy) = \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & \lambda \\ 0 & 0 & 0 \end{pmatrix} = F(\begin{pmatrix} 0 & 0 & \alpha\sigma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. After simple calculate on the right side, we deduce F is a (multiplicative) homogeneralized b-derivation of R.

The Results

Note: For any $x, y \in R$ the symbol $[x, y]$ represents the Lie commutator $xy - yx$ and the Jordan product $x \circ y = xy + yx$.

Theorem (1)

Let R be a semiprime ring and K be a nonzero dense ideal of R. Suppose $F\colon R \to Q_{mr}$ is a multiplicative b-homogeneralized derivation associated with derivation $d\colon R \to R$ satisfying the condition $[F(\sigma), \tau] \in Z(R)$ for all $\sigma, \tau \in K$ and any $0 \neq b \in Q_s \subseteq Q_{mr}$. Then either d is commuting over R or $[\sigma, b] = 0$.

Corollary (2)

Let R be a semiprime ring and K be a nonzero dense ideal of R. Suppose $F\colon R \to Q_{mr}$ is a multiplicative b-homogeneralized derivation associated with derivation $d\colon R \to R$ satisfying the condition $F(\sigma) \circ \tau \in Z(R)$ for all $\sigma, \tau \in K$ and any $0 \neq b \in Q_s \subseteq Q_{mr}$. Then either d is commuting over R or $[\sigma, b] = 0$.

Proposition (3)

Suppose R is ring. If $F\colon R \to R$ acts as a multiplicative b-homogeneralized derivation associated with derivation $d\colon R \to R$ over R. For any $0 \neq \sigma \in Z(R)$ yields $F(\sigma) \in Z(R)$.

Theorem (4)

Let R be a semiprime ring and K be a nonzero dense ideal of R. Assume $F\colon R \to Q_{mr}$ is a multiplicative b-homogeneralized derivation associated with derivation $d\colon R \to R$ such that $F([\sigma, \tau]) = 0$ for all $\sigma, \tau \in K$ and any $0 \neq b \in Q_s \subseteq Q_{mr}$. Then either d is commuting over R or $[\sigma, b] = 0$.

The above theorem supplies the attached corollary:

Corollary (5)

Let R be a semiprime ring and K be a nonzero dense maximal ideal of R. Suppose $F\colon R \to Q_{mr}$ is a multiplicative b-homogeneralized derivation associated with derivation $d\colon R \to R$ satisfying the identity $[F(\sigma), \tau] = 0$ for all $\sigma, \tau \in K$ and any $0 \neq b \in Q_s \subset Q_{mr}$. Then either b acts as centralizer of K or $d(K) = 0$.

Theorem (6)

Suppose R is a semiprime ring and K is a nonzero dense maximal ideal of R. If $F\colon R \to Q_{mr}$ is a multiplicative b-homogeneralized derivation associated with derivation $d\colon R \to R$ when $F(\sigma\tau) \mp \sigma\tau \in Z(R)$ for all $\sigma, \tau \in K$ and any $0 \neq b \in Q_s \subset Q_{mr}$. Then either b acts as centralizer of K or $d(K) = 0$.

Theorem (7)

Suppose that R is a prime ring and K is a nonzero dense maximal ideal of R. If $F\colon R \to Q_{mr}$ is a multiplicative b-homogeneralized derivation associated with derivation $d\colon R \to R$. Then $b \in Z(R)$ when

(i) $[F(\sigma), \tau] \in Z(R)$,

(ii) $F(\sigma\tau) \mp \sigma\tau \in Z(R)$,

(iii) $F(\sigma \circ \tau) = 0$,

(v) $F([\sigma, \tau]) = 0$

for all $\sigma, \tau \in K$ and any $0 \neq b \in Q_s \subset Q_{mr}$.

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Thank you

Questions?

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