Stability of vertical minimal surfaces in three-dimensional sub-Riemannian manifolds

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A *sub-Riemannian manifold* is a smooth manifold M together with a completely non-integrable smooth distribution $\mathcal{H}$ on M (it is called a *horizontal distribution*) and a smooth field of Euclidean scalar products $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ on $\mathcal{H}$ (it is called a *sub-Riemannian metric*). In particular, $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ can be constructed as the restriction of some Riemannian metric $\langle \cdot, \cdot \rangle$ on M to $\mathcal{H}$. Here we will assume that all sub-Riemannian structures are of this form.

Let Σ be a smooth oriented surface in a three-dimensional sub-Riemannian manifold M. If N_h is the orthogonal projection of the unit normal field N of Σ (in the Riemannian sense) onto $\mathcal{H}$ and $d\Sigma$ is the Riemannian area form of Σ, then the *sub-Riemannian area* of a domain $D \subset \Sigma$ is defined as

$$A(D) = \int_D |N_h| \, d\Sigma.$$

The *normal variation* of the surface Σ defined by a smooth function u is the map

$$\varphi : \Sigma \times I \to M : \varphi_s(p) = exp_p(su(p)N(p)),$$

where I is an open neighborhood of 0 in $\mathbb{R}$ and exp_p is the Riemannian exponential map at p. In other words, we construct the variation in the traditional Riemannian way by drawing the geodesic through each point $p \in \Sigma$ in the direction of the normal vector $u(p)N(p)$.

Denote

$$A(s) = \int_{\Sigma_s} |N_h| \, d\Sigma_s,$$

where $\Sigma_s = \varphi_s(\Sigma)$. Then $A'(0)$ is called the first *(normal) area variation* defined by φ, and $A''(0)$ is called the *second* one.
A surface Σ is called *minimal* if $A'(0) = 0$ for any normal variations with compact support in $\Sigma \setminus \Sigma_0$, where $\Sigma_0 = \{p \in \Sigma \mid N_h(p) = 0\}$ is the *singular set* of Σ. Note that here we also follow the Riemannian tradition by defining minimal surfaces as stationary points of the sub-Riemannian area functional.
A minimal surface Σ is called *stable* if $A''(0) \geqslant 0$ for any normal variations with compact support in $\Sigma \setminus \Sigma_0$.

We will call a surface Σ in a three-dimensional sub-Riemannian manifold *vertical* if $T_p\Sigma \perp \mathcal{H}_p$ for each $p \in \Sigma$, that is, $N \in \mathcal{H}$. In particular, for such surfaces $N_h = N$ and $\Sigma_0 = \varnothing$.

Proposition

Let Σ be a vertical surface in a three-dimensional sub-Riemannian manifold M. Then its first normal area variation defined by a smooth function u with compact support equals

$$A'(0) = -\int_{\Sigma} 2Hu \, d\Sigma,$$

where H is the Riemannian mean curvature of Σ.

Corollary

A vertical surface is minimal in the sub-Riemannian sense if and only if in is minimal in the Riemannian sence.

Proposition

Let Σ be a vertical minimal surface in a three-dimensional sub-Riemannian manifold M. Then its second normal area variation defined by a smooth function u with compact support equals

$$A''(0) = \int_{\Sigma} -\left(X(u) - \langle \nabla_N X, N \rangle u\right)^2 + + \underbrace{|\nabla_\Sigma u|^2 - \left(\mathrm{Ric}\,(N,N) + |B|^2\right) u^2}_{\text{The Riemannian part}} \, d\Sigma,$$

where ∇ and Ric are the Riemannian connection and the Ricci tensor of M respectively, X is the unit normal vector field of $\mathcal{H}$ (which is tangent to Σ because it is vertical), ∇_Σ and $|B|$ are the Riemannian gradient and the norm of the second fundamental form of Σ respectively.

Corollary

If a vertical minimal surface is stable in the sub-Riemannian sense, it is also stable in the Riemannian sense.

Let us look at some examples.
In [3] A. Hurtado and C. Rosales considered the standard three-dimensional sphere $(S^3, \langle \cdot, \cdot \rangle)$ embedded in the Euclidean space $\mathbb{R}^4$ with coordinates (x, y, z, w) with the horizontal distribution orthogonal to the Hopf field

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - w\frac{\partial}{\partial z} + z\frac{\partial}{\partial w}$$

(completely non-integrable and left-invariant with respect to the Lie group structure of S^3) and showed that complete connected vertical minimal surfaces are Clifford tori. It is well-known that they are not stable in the Riemannian sense, hence also in the sub-Riemannian sense.

The three-dimensional Heisenberg group $\mathbb{H}^1$ (also known as the three-dimensional Thurston geometry *Nil*) is the space $\mathbb{R}^3$ with coordinates (x, y, z) and with the following orthonormal basis of left-invariant vector fields defined by its nilpotent Lie group structure:

$$X_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, X_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, X_3 = \frac{\partial}{\partial z}.$$

Let the horizontal distribution $\mathcal{H}$ be orthogonal to X_3. It is completely non-integrable because $[X_1, X_2] = X_3$.
In [2] it was shown by A. Hurtado, M. Ritoré and C. Rosales that a complete connected minimal surface with the empty singular set (in particular, vertical) in $\mathbb{H}^1$ is stable if and only if it is a vertical (that is, parallel to $X_3 = \frac{\partial}{\partial z}$) Euclidean plane.
Note that there are no other vertical minimal surfaces in $\mathbb{H}^1$.

The manifold $\widetilde{E(2)}$ is the universal covering of the proper motions group of the Euclidean plane. This is the space $\mathbb{R}^3$ with coordinates (x, y, z) and with the following orthonormal basis of left-invariant vector fields defined by its solvable Lie group structure:

$$X_1 = \cos z \frac{\partial}{\partial x} + \sin z \frac{\partial}{\partial y}, X_2 = -\sin z \frac{\partial}{\partial x} + \cos z \frac{\partial}{\partial y}, X_3 = \frac{\partial}{\partial z}.$$

Note that its Riemannian metric is Euclidean and that $[X_3, X_1] = X_2$, so the horizontal distribution $\mathcal{H}$ orthogonal to X_2 is completely non-integrable.
In [1] we proved that all complete connected vertical minimal surfaces in $\widetilde{E(2)}$ are Euclidean planes $z = C$ and standard helicoids. We showed that planes are stable in the sub-Riemannian sense, and it is known that helicoids are not stable in the Riemannian sense, hence also in the sub-Riemannian sense.

The three-dimensional Thurston geometry *Sol* is the space $\mathbb{R}^3$ with coordinates (x, y, z) and with the following orthonormal basis of left-invariant vector fields defined by its solvable Lie group structure:

$$X_1 = \frac{1}{\sqrt{2}}\left(e^{-z}\frac{\partial}{\partial x} + e^{z}\frac{\partial}{\partial y}\right), X_2 = \frac{1}{\sqrt{2}}\left(e^{-z}\frac{\partial}{\partial x} - e^{z}\frac{\partial}{\partial y}\right),$$

$$X_3 = \frac{\partial}{\partial z}.$$

Note that $[X_2, X_3] = X_1$, so the left-invariant distribution $\mathcal{H}$ orthogonal to X_1 is completely non-integrable. Let us consider a sub-Riemannian structure on *Sol* such that $\mathcal{H}$ is horizontal.

It follows from the results of L. Masaltsev in [4] that any complete connected vertical minimal surface in *Sol* is either a Euclidean plane $z = C$ or a "helicoid"

$$(s,t) \mapsto \left(\frac{1}{\sqrt{2}} e^{-t} s + C_1, \frac{1}{\sqrt{2}} e^{t} s + C_2, t \right).$$

Using this description, we are able to prove the following.

Proposition

All vertical minimal surfaces in Sol are stable in the sub-Riemannian sense and thus in the Riemannian sense.

The three-dimensional Thurston geometry $\widetilde{SL(2,\mathbb{R})}$ can be described as the universal covering of the unit tangent bundle of the hyperbolic plane H^2 with the Sasaki metric, that is, the half-space $\{(x,y,z) \in \mathbb{R}^3 \mid y > 0\}$ with the following orthonormal basis of left-invariant vector fields with respect to its simple Lie group structure:

$$X_1 = y\left(-\sin z \frac{\partial}{\partial x} + \cos z \frac{\partial}{\partial y}\right) + \sin z \frac{\partial}{\partial z},$$

$$X_2 = y\left(-\cos z \frac{\partial}{\partial x} - \sin z \frac{\partial}{\partial y}\right) + \cos z \frac{\partial}{\partial z}, X_3 = \frac{\partial}{\partial z}.$$

In particular, $[X_1, X_2] = -X_3$, so the left-invariant distribution $\mathcal{H}$ orthogonal to X_3 is completely non-integrable. Consider a sub-Riemannian structure on this manifold such that $\mathcal{H}$ is horizontal.

We than obtain the following description.

Theorem

Any complete connected vertical minimal surface in $\widetilde{SL(2,\mathbb{R})}$ has either the parameterization $(s,t) \mapsto (C, s, t)$ or $(s,t) \mapsto \left(C_1 + \frac{1}{C_2}\sin C_2 s, -\frac{1}{C_2}\cos C_2 s, t\right)$ and so is a cylinder over a geodesic in H^2.

All vertical minimal surfaces in $\widetilde{SL(2,\mathbb{R})}$ are stable in the sub-Riemannian sense and thus in the Riemannian sense.

Thank you!

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