Disjoint dynamical properties of wedge operators

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S. Ivković, Hypercyclic operators on Hilbert C*-modules, Filomat **38** (2024), 1901–1913.

Definition

Let $N \geq 2$, and $T_1, T_2, \ldots, T_N$ be bounded linear operators acting on a separable Banach space $\mathcal{X}$.

1. The finite sequence $T_1, T_2, \ldots, T_N$ is called *disjoint hypercyclic* or simply *d-hypercyclic* if there exists an element $x \in \mathcal{X}$ such that the set

$$\{(x, x, \ldots, x), (T_1 x, T_2 x, \ldots, T_N x), (T_1^2 x, T_2^2 x, \ldots, T_N^2 x), \ldots\} \quad (1)$$

   is dense in $\mathcal{X}^N$. In this case, the element x is called a *d-hypercyclic vector*. If the set of all d-hypercyclic vectors for $T_1, T_2, \ldots, T_N$ is dense in $\mathcal{X}$, then we say that $T_1, T_2, \ldots, T_N$ are *densely d-hypercyclic* .

2. The finite sequence $T_1, T_2, \ldots, T_N$ is called *disjoint topologically transitive* or simply *d-topologically transitive* if for any non-empty open subsets $U, V_1, \ldots, V_N$ of $\mathcal{X}$, there exist a natural number $n \in \mathbb{N}$ such that

$$U \cap T_1^{-n}(V_1) \cap \cdots \cap T_N^{-n}(V_N) \neq \varnothing. \quad (2)$$

Definition

Let $\{n_k\}_k$ be a strictly increasing sequence of positive integers. We say that $T_1, \ldots, T_N \in B(\mathcal{X})$ satisfy the *d-hypercyclicity criterion with respect to* $\{n_k\}_k$ whenever there exist some dense subsets $\mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_N$, of $\mathcal{X}$ and mappings $S_{l,k} : \mathcal{X}_l \to \mathcal{X}$ $(1 \leq l \leq N, k \in \mathbb{N})$ such that

$$T_l^{n_k} \to 0 \text{ pointwise on } \mathcal{X}_0,$$

$$S_{l,k} \to 0 \text{ pointwise on } \mathcal{X}_l \text{ and}$$

$$(T_l^{n_k} S_{i,k} - \delta_{i,l} Id_{\mathcal{X}_l}) \to 0 \text{ pointwise on } \mathcal{X}_l (1 \leq i, l \leq N) \qquad (3)$$

as $k \to \infty$. Also, we say that $T_1, \ldots, T_N$ satisfy the *d-hypercyclicity criterion* if there exists some sequence $\{n_k\}_k$ for which (3) is satisfied. If $T_1, \ldots, T_N$ satisfy the d-hypercyclicity criterion, then they are densely disjoint hypercyclic, so d-hypercyclicity criterion is stronger than dense disjoint hypercyclicity.

In this presentation, we assume that $\mathcal{H}$ is a separable Hilbert space with an orthonormal basis $\{e_j\}_{j\in\mathbb{Z}}$. For each $m \in \mathbb{N}$, we set $L_m := Span\{e_{-m}, e_{-m+1}, \ldots, e_{m-1}, e_m\}.$, and we let P_m be the orthogonal projection onto L_m.
The set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$ is denoted by $B(\mathcal{H})$. Also, the set of all compact (finite rank, respectively) elements of $B(\mathcal{H})$ is denoted by $B_0(\mathcal{H})$ ($B_{00}(\mathcal{H})$, respectively).

Definition

Let $U, W \in B(\mathcal{H})$. We define the operator $T_{U,W} : B(\mathcal{H}) \to B(\mathcal{H})$ by

$$T_{U,W}(F) := WFU \tag{4}$$

for all $F \in B(\mathcal{H})$.

Theorem

Let $W_1, \ldots, W_N$ be invertible bounded linear operators on $\mathcal{H}$. Let $U \in B(\mathcal{H})$ be a unitary operator in $B(\mathcal{H})$ such that for each $k \in \mathbb{N}$ there exists an $N_k \in \mathbb{N}$ with

$$U^n(L_k) \perp L_k \quad \text{for all } n \geq N_k. \tag{5}$$

For each $k \in \mathbb{N}$ denote the operator T_{U,W_k} on $B_0(\mathcal{H})$ by T_k. Also, assume that $\{r_k\}_{k=1}^{N} \subseteq \mathbb{N}$ such that $0 < r_1 < r_2 < \ldots < r_N$. Then, the following conditions are equivalent.

(i) The set of all d-hypercyclic vectors of $T_1^{r_1}, \dots, T_N^{r_N}$ is dense in $B_0(\mathcal{H})$.

(ii) For each $m \in \mathbb{N}$ there exist sequences $\{D_k\}_{k=1}^{\infty}, \{G_k^{(1)}\}_{k=1}^{\infty}, \dots, \{G_k^{(N)}\}_{k=1}^{\infty}$ of operators in $B_0(\mathcal{H})$, and a strictly increasing sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$ such that for each $l \in \{1, \dots, N\}$,

$$\lim_{k\to\infty} \left\| D_k - P_m \right\| = \lim_{k\to\infty} \left\| G_k^{(l)} - P_m \right\| = 0, \tag{6}$$

$$\lim_{k\to\infty} \left\| W_l^{r_k n_k} D_k \right\| = \lim_{k\to\infty} \left\| W_l^{-r_k n_k} G_k^{(l)} \right\| = 0, \tag{7}$$

and, for each pair of distinct $s,\ l \in \{1, \dots, N\}$,

$$\lim_{k\longrightarrow\infty} \left\| W_l^{r_l n_k} W_s^{-r_s n_k} G_k^{(s)} \right\| = 0. \tag{8}$$

Remark

One can prove a similar result for the operator $F \mapsto UFW$, where W is invertible and U is a unitary operator satisfying the condition (5). For this, it would be enough to replace the relations (7) and (8) by the conditions

$$\lim_{k\to\infty} \left\| D_k W_l^{r_k n_k} \right\| = \lim_{k\to\infty} \left\| G_k^{(l)} W_l^{-r_k n_k} \right\| = 0$$

and

$$\lim_{k\longrightarrow\infty} \left\| G_k^{(s)} W_s^{-r_s n_k} W_l^{r_l n_k} \right\| = 0,$$

respectively. It follows by passing to the adjoints that if $W_1, \ldots, W_N$ satisfy the conditions (6), (7) and (8), then $W_1^*, \ldots, W_N^*$ satisfy these new conditions.

Example

Assume that α is a translation on $\mathbb{Z}$. Set $U_\alpha(e_j) := e_{\alpha(j)}$ for all $j \in \mathbb{Z}$. Then, U_α is a unitary operator on H satisfying the property (5).

Theorem

The following statements are equivalent:

1) The operators $T^{r_1}_{\tilde{U},W_1}, \ldots, T^{r_N}_{\tilde{U},W_N}$ *satisfy* $d-$*hypercyclicity criterion on* $B_0(H)$, *where* $\tilde{U}$ *is a unitary operator on* H *satisfying the condition* (5).

2) The operators $T^{r_1}_{U,W_1}, \ldots, T^{r_N}_{U,W_N}$ *satisfy* $d-$*hypercyclicity criterion on* $B_0(H)$ *for every unitary operator* U.

3) The operators $W_1^{r_1}, \ldots, W_N^{r_N}$ *satisfy* $d-$*hypercyclicity criterion on* H.

The similar statements hold if we consider $B_1(H)$ *or* $B_2(H)$ *instead of* $B_0(H)$.

Example

Let $r_1 \in \mathbb{N}$ and $r_2 = 2r_1$. Put W_1 and W_2 to be the operators on H defined as

$$W_1(e_j) = \begin{cases} 2e_{j+1} & \text{for } j < 0, \\ \frac{1}{2} e_{j+1} & \text{for } j \geq 0; \end{cases}$$

$$W_2(e_j) = \begin{cases} 3e_{j+1} & \text{for } j < 0, \\ \frac{1}{3} e_{j+1} & \text{for } j \geq 0. \end{cases}$$

Then W_1 and W_2 are bounded operators and $T^{r_1}_{U,W_1}$ and $T^{r_2}_{U,W_2}$ satisfy $d-$hypercyclicity criterion for every unitary operator U.

In general, consider $\ell_2(\mathbb{Z})$ and let $\{z_j\}$ be the natural orthonormal basis for $\ell_2(\mathbb{Z})$. Let V be the unitary operator from $\ell_2(\mathbb{Z})$ onto H given by $Vz_j = e_j$ for all $j \in \mathbb{Z}$. By the previous arguments it follows that if $\tilde{T}_1^{\,r_1}, \ldots, \tilde{T}_N^{\,r_N}$ are disjoint hypercyclic weighted shifts on $\ell_2(\mathbb{Z})$, then $T_1^{r_1}, \ldots, T_N^{r_N}$ satisfy $d-$hypercyclicity criterion on $B_0(H)$ where $T_i^{r_i}(F) = V\tilde{T}_i^{\,r_i}V^*FU^{r_i}$ for all $i \in \{1, \ldots, N\}$ and $F \in B_0(H)$ and U is an arbitrary unitary operator on H. For more details about disjoint hypercyclic weighted shifts, see [**bms14**], [**bp07**].

Example

Let $H = L^2(\mathbb{R})$, $\alpha(t) = t - 1$ for all $t \in \mathbb{R}$,
$w_1 = 2\mathcal{X}_{\mathbb{R}^-} + \frac{1}{2}\mathcal{X}_{\mathbb{R}^+}$ and $w_2 = 3\mathcal{X}_{\mathbb{R}^-} + \frac{1}{3}\mathcal{X}_{\mathbb{R}^+}$.

Choose an $r_1 \in \mathbb{N}$ and set $r_2 = 2r_1$.

Let $W_1, W_2 \in B(H)$ be given by $W_j(f) = w_j \cdot (f \circ \alpha)$ for all $f \in H$ and $j \in \{1, 2\}$.

Then $T^{r_1}_{U,W_1}$ and $T^{r_2}_{U,W_2}$ satisfy $d-$hypercyclicity criterion for every unitary operator U.

In general, if α is a translation on $\mathbb{R}$ and $w_1, \ldots, w_N$ are positive, measurable, bounded weight functions satisfying that $w_1^{-1}, \ldots, w_N^{-1}$ are also bounded, then we can consider the corresponding sequence $W_1^{r_1}, \ldots, W_N^{r_N}$ of weighted translation operators on $L^2(\mathbb{R})$.
If for every $l, s \in \{1, \ldots, N\}$ and $m \in \mathbb{N}$ we have that

$$\lim_{n\to\infty} \sup_{t\in[-m,m]} \mid \prod_{j=0}^{r_l n-1} (w_l \circ \alpha^{j-r_l n})(t) \mid = \lim_{n\to\infty} \sup_{t\in[-m,m]} \mid \prod_{j=0}^{r_l n-1} (w_l \circ \alpha^{j})^{-1}(t) \mid = 0$$

and in addition

$$\lim_{n\to\infty} \sup_{t\in[-m,m]} \frac{|\prod_{j=1}^{r_l n} (w_l \circ \alpha^{r_s n-j})(t)|}{|\prod_{j=0}^{r_s n-1} (w_s \circ \alpha^{j})(t)|} = 0,$$

then the operators $W_1^{r_1}, \ldots, W_N^{r_N}$ satisfy the conditions (6), (7) and (8) .

For an operator T in $B(H)$, we will denote the left an the right multiplier by L_T and R_T, respectively.

Corollary

Let $W_1, \ldots, W_N$ be invertible bounded linear operators on H and $\{r_k\}_{1 \leq k \leq N} \subseteq \mathbb{N}$ such that $0 \leq r_1 \leq \cdots \leq r_N$. Then $L_{W_1^{r_1}}, \ldots, L_{W_N^{r_N}}$ satisfy $d-$hypercyclicity criterion on $B_0(H)$ if and only if $W_1^{r_1}, \ldots, W_N^{r_N}$ satisfy $d-$hypercyclicity criterion on H. The similar statements hold if we replace $B_0(H)$ by $B_1(H)$ or $B_2(H)$.

Let $(B(H), SOT)$ denote the space $B(H)$ equipped with the strong operator topology.

Corollary

We have (ii) *implies* (i).
(i) $T^{r_1}_{U,W_1}, \ldots, T^{r_N}_{U,W_N}$ *are disjoint topologically transitive in* $(B(H), SOT)$ *for every unitary operator* U.
(ii) *For each* $m \in \mathbb{N}$ *there exist sequences* $\{D_k\}_{k=1}^{\infty}, \{G_k^{(1)}\}_{k=1}^{\infty}, \ldots, \{G_k^{(N)}\}_{k=1}^{\infty}$ *of operators in* $B(H)$ *and a strictly increasing sequence* $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$ *such that for each* $l \in \{1, \ldots N\}$,

$$s - \lim_{n\to\infty} D_k = s - \lim_{n\to\infty} G_k^{(l)} = P_m$$

and the conditions (7) *and* (8) *hold.*
In particular, if $W_1, \ldots, W_N$ *satisfy the conditions* (6), (7) *and* (8), *then* $T^{r_1}_{U,W_1}, \ldots, T^{r_N}_{U,W_N}$ *are disjoint topologically transitive in* $(B(H), SOT)$ *for every unitary operator* U.

Let $\mathcal{A}$ be a non-unital C^*-algebra such that $\mathcal{A}$ is a closed two-sided ideal in a unital C^*-algebra $\mathcal{A}_1$. Let Φ be an isometric $*$-isomorphism of $\mathcal{A}_1$ such that $\Phi(\mathcal{A}) = \mathcal{A}$. Assume that there exists a net $\{p_\alpha\}_\alpha \subseteq \mathcal{A}$ consisting of self-adjoint elements with $\| p_\alpha \| \leq 1$ for all α and such that $\{p_\alpha^2\}_\alpha$ is an approximate unit for $\mathcal{A}$. Suppose in addition that for all α there exists some $N_\alpha \in \mathbb{N}$ such that $\Phi^n(p_\alpha) \cdot p_\alpha = 0$ for all $n \geq N_\alpha$.

Let $b \in G(\mathcal{A}_1)$ and $T_{\Phi,b}$ be the operator on $\mathcal{A}_1$ defined by $T_{\Phi,b}(a) = b \cdot \Phi(a)$ for all $a \in \mathcal{A}_1$. Then $T_{\Phi,b}$ is a bounded linear operator on $\mathcal{A}_1$ and since $\mathcal{A}$ is an ideal in $\mathcal{A}_1$, it follows that $T_{\Phi,b}(\mathcal{A}) \subseteq \mathcal{A}$ because $\Phi(\mathcal{A}) = \mathcal{A}$.

Theorem

The following statements are equivalent.
(i) $T_{\Phi,b}$ *is hypercyclic on* $\mathcal{A}$.
(ii) For every p_α *there exists a strictly increasing sequence* $\{n_k\}_k \subseteq \mathbb{N}$ *and sequences* $\{q_k\}_k$, $\{d_k\}_k$ *in* $\mathcal{A}$ *such that*

$$\lim_{k\to\infty} \| q_k - p_\alpha^2 \| = \| d_k - p_\alpha^2 \| = 0$$

and

$$\lim_{k\to\infty} \| \Phi^{-n_k}(b)\Phi^{-n_k+1}(b)\ldots\Phi^{-1}(b)q_k \|$$
$$= \lim_{k\to\infty} \| \Phi^{n_k-1}(b^{-1})\Phi^{n_k-2}(b^{-1})\ldots\Phi(b^{-1})b^{-1}d_k \| = 0$$

If $a \in \mathcal{A}_1$, in the sequel we shall denote by L_a the left multiplier by a.

Corollary

If there exist dense subsets Ω_1 and Ω_2 of $\mathcal{A}$ and a strictly increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ such that

$$L_{\Phi^{-n_k}(b)\Phi^{-n_k+1}(b)\ldots\Phi^{-1}(b)} \overset{k\to\infty}{\longrightarrow} 0$$

pointwise on Ω_1 and

$$L_{\Phi^{n_k-1}(b^{-1})\Phi^{n_k-2}(b^{-1})\ldots\Phi(b^{-1})b^{-1}} \overset{k\to\infty}{\longrightarrow} 0$$

pointwise on Ω_2, then $T_{\Phi,b}$ is hypercyclic on $\mathcal{A}$.

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