Sobolev embedding and quality of its non-compactness

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Introduction

$T : X \to Y$ is bounded linear map between Banach spaces X and Y and B_X is unit ball in X.

Entropy numbers:

$$e_k(T) := \inf \left\{ \varepsilon > 0 : T(B_X) \text{ can be covered by } 2^{k-1} \text{ balls in } Y \text{ with radius } \varepsilon \right\}$$

s-Numbers and n-Widths:

$a_n(T) := \inf_{P_n} \sup_{y \in T(B_X)} \|y - P_n(y)\|_Y$ **(Approx. numbers)**
where $P_n \in L(X, Y)$ with rank $<$ n.

$d_n(T) := \inf_{Y_n} \sup_{z \in T(B_X)} \inf_{y \in Y_n} \|y - z\|_Y$ **(Kolmogorov numbers)**
where $Y_n \subset Y$ is n-dimensional subspace.

$c_n(T) := \inf_{L_n} \sup_{y \in T(B_X) \cap L_N} \|y\|_Y$ **(Gelfand numbers)**
where L_n are closed subspaces of Y with codimension at most n.

$b_n(T) := \sup_{Y_n} \sup\{\lambda \geq 0 : Y_n \cap \lambda B_Y \subset T(B_Y)\}$ **(Bernstein numbers)**
where Y_n is a subset of Y with dimension n.

$i_n(T) := \sup \left\{ \|A\|^{-1} \|B\|^{-1} \right\}$ **(isomorphism numbers)**
where the sup. is taken over all Banach spaces G with $dim(G) \geq n$ and maps $A \in L(Y, G)$ and $B \in L(G, X)$ such that ATB is identity on G.

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We have much more s-Numbers and n-Widths like:
$m_n(T)$ - Mityagin numbers, $n_n(T)$ - Weyl numbers
$y_n(T)$ - Chang numbers, $h_n(T)$ - Hilbert numbers, ...

For every s-number we have: $s_1 = \|T\| \geq s_2 \geq ... \geq 0$ + other properties
...

Above mentioned s-numbers are related:
$a_n(T) \geq max(c_n(T), d_n(T)) \geq min(c_n(T), d_n(T))$
$\geq max(b_n(T)m_n(T)) \geq min(b_n(T), m_n(T)) \geq i_n(T) \geq h_n(T)$

There are many duality relations like: $a_n(T') \leq a_n(T) \leq 5a_n(T')$,
$c_n(T) = d_n(T')$, $m_n(T) = b_n(T')$, ...

T - compact iff $lim_{n\to\infty} e_n(T) = 0$ iff $\lim_{n\to\infty} d_n(T) = 0$.
Measure of non-compactness: $\beta(T) = \lim e_n(T)$,
plainly $0 \leq \beta(T) \leq \|T\|$

We say that T is **maximally noncompact** if $\|T\| = \beta(T)$.

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Strictly singular maps

Let X, Y be Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$ respectively. The map $T : X \to Y$ is said to be *strictly singular* if there is no infinite dimensional closed subspace Z of X such that the restriction $T|_Z$ of T to Z is an isomorphism of Z onto $T(Z)$.

Equivalently, for each infinite-dimensional closed subspace Z of X,

$$\inf\{\|Tx\|_Y : \|x\|_X = 1, x \in Z\} = 0.$$

If T has the property that given any $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that if E is a subspace of X with $\dim E \geq N(\varepsilon)$, then there exists $x \in E$, with $\|x\|_X = 1$, such that $\|Tx\|_Y \leq \varepsilon$, then T is said to be *finitely strictly singular.*

This second definition can be expressed in terms of the Bernstein numbers $b_k(T)$ of T. We recall that these are given, for each $k \in \mathbb{N}$, by

$$b_k(T) = \sup_{E \subset X, \dim E = k} \; \inf_{x \in E, \|x\|_X = 1} \|Tx\|_Y .$$

Then T is finitely strictly singular if and only if

$$b_k(T) \to 0 \text{ as } k \to \infty.$$

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The relations between these notions and that of compactness of T are illustrated by the following diagram:

$$T \text{ compact} \implies T \text{ finitely strictly singular} \implies T \text{ strictly singular}$$

and each reverse implication is false in general.

Sobolev Embedding

If T is an embedding map between function spaces on an open set $\Omega \subset \mathbf{R}^n$, possible reasons for noncompactness include:
(i) Ω unbounded
(ii) if Ω bounded then due *bad* boundary $\partial\Omega$,or
(iii) due particular values of the parameters involved in functions spaces (inner structure of spaces)

Sobolev Embedding: We consider: $id : W_0^{k,p}(\Omega) \to L^q(\Omega)$ with $k \in \mathbf{N}$, $p \in [1, \infty)$, $kp < n$, $1 \leq q < np/(n - kp)$.

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Sobolev Embedding - case (i)

Question: Let $n = 2$, $\Omega = \mathbf{R} \times (0, \pi)$ and $I : W_0^{1,p}(\Omega) \to L^p(\Omega)$. We can see that I is noncompact and that $\beta(I) > 0$. What is the exact value of $\beta(I)$?

Answer:(Edmunds, Mihula, L, 21) Let $n \geq 2$, $k \in \{1, ..., n-1\}$, $1 < p < \infty$ and $-\infty < a_i < b_i < \infty$. Set $D = \mathbf{R}^k \times \Pi_{i=1}^{n-k}(a_i, b_i)$; the norm on $W_0^{1,p}(D)$ is defined by:

$$\left(\|u\|_{p,D}^p + \||\nabla u|_{l^p}\|_{p,D}^p\right)^{1/p}.$$

Consider $I_p : W_0^{1,p}(D) \to L^p(D)$. Then

$$\beta(I_p) = \|I_p\| = \left(1 + (p-1)\left(\frac{2\pi}{p\sin(\pi/n)}\right)^p \sum_{i=1}^{n-k}(b_i - a_i)^{-p}\right)^{-1/p}$$

Note: For $p = 2$, $n = 2$, $b_1 - a_1 = \pi$ we have $\beta(I) = \|I\| = 1/\sqrt{2}$.

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By product: Set $R = \Pi_{i=1}^n (a_i, b_i)$. Note that the extreme function for Rayleigh quotient

$$\inf_{0 \neq u \in W_0^{1,p}(R)} \frac{\| |\operatorname{grad} u|_{l^p} \|_{p,R}^p}{\|u\|_{p,R}^p}$$

is the first eigenvalue of the pseudo-p-Laplacian operator with Dirichlet conditions, i.e.: $\tilde{\Delta}_p u = \tilde{\lambda}_p |u|^{p-2} u$, with $u = 0$ on ∂R, where

$$\tilde{\Delta}_p u = \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right).$$

And the first eigenfunction is $u(x) = \Pi_{i=1}^n \sin_p \left(\frac{\pi_p (x_i - a_i)}{b_i - a_i} \right), x \in R$.
Also this function is the extreme function for Sobolev embedding: $I : W_0^{1,p}(R) \to L^p(R)$.
More-over functions of the form $\Pi_{i=1}^n \sin_p \left(\frac{\pi_p k_i (x_i - a_i)}{b_i - a_i} \right), x \in R$, and $k_i \in \mathbf{N}$ are eigenfunctions of the above pseudo-p-Laplacian.
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Sobolev Embedding - case (ii)

When $\Omega \subset \mathbb{R}^n$ is bounded and has a "good" boundary then, obviously, $E : W_p^1(\Omega) \to L_p(\Omega)$ is compact.

Theorem (Edmunds , L. 22)

Let $n \geq 2$. There is a bounded open set $\Omega \subset \mathbb{R}^n$ such that $E : W_p^1(\Omega) \to L_p(\Omega)$ is not strictly singular.

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Sobolev Embedding - case (ii)

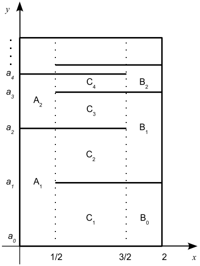


Figure: The domain Ω_1

Set $a_i = \sum_{k=1}^{i} k^{-p} \ (i \in \mathbb{N})$, $a_0 = 0$, and $\Omega_{b,m} = \Omega_b \cap ([0, 2b] \times [0, a_m])$ Now we construct a continuous function $f_{b,m} : \Omega_{b,m} \to \mathbb{R}$ that has the shape of an increasing staircase with slope $1/b$ on C_i and landings on A_i and B_i with zero value at B_0. More precisely we can write that:

$$f_{b,m}(x) = \begin{cases} 0, & x \in B_0 \cup C_1 \cup A_1, \\ 2i - 2, & x \in A_i, (i \in \mathbb{N}) \\ 2i - 1, & x \in B_i (i \in \mathbb{N}) \end{cases}$$

Sobolev Embedding - case (ii)

A routine calculations show that

$$\|\nabla f_{b,m}\|_{p,\Omega_{b,m}} = \left(\sum_{i=1}^{m} |C_i|\right)^{1/p} b^{-1} = b^{-(p-1)/p} \left(a_m\right)^{1/p},$$

$$\|f_{b,m}\|_{p,\Omega_{b,m}} \approx \left(\sum_{i=1}^{[m/2]} \left\{(2i-1)^{-p} + (2i)^{-p}\right\} i^p\right)^{1/p} b^{1/p} \approx \left(\sum_{i=1}^{[m/2]} 1\right)^{1/p} b^{1/p}$$

$$= [m/2]^{1/p} b^{1/p}, \quad \text{where } [.] \text{ is the greatest integer function.}$$

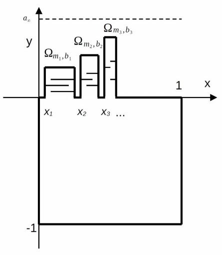
Thus

$$\sup_{g \in W_p^1(\Omega_{b,m})} \frac{\|g\|_{p,\Omega_{b,m}}}{\|\nabla g\|_{p,\Omega_{b,m}}} \gtrapprox \left(\frac{[m/2]b}{a_m}\right)^{1/p}.$$

Sobolev Embedding - case (ii)

Now we set

$$\Omega := ((0,1)\times(-1,0)) \cup \left(\cup_{i=1}^{\infty}\left((\Omega_{b_i,m_i} \cup (0,2b_i)\times\{0\}) + (x_i,0)\right)\right).$$



To justify this, consider the sequence $\{f_i\}$ of functions defined by $f_i(x) = f_{b_i,m_i}(x - \tilde{x}_i)$, where $\tilde{x}_i = (x_i, 0)$. Then $\operatorname{supp} f_i \subset \overline{\Omega_{b_i,m_i} + \tilde{x}_i}$ and

$$\frac{\|f_i\|_{p,\Omega}}{\|\nabla f_i\|_{p,\Omega}} \gtrsim \gamma.$$

The claim follows.

Sobolev Embedding - case (iii)

Let $k, n \in \mathbf{N}$, $k < n$, Ω open subset in $\mathbf{R}^n$, $p \in [1, n/k)$ and $p^* = \frac{np}{n-kp}$ then one has

$$I_1 : V_0^{k,p} \to L^{p^*}(\Omega)$$

where $\|u\|_{V_0^{k,p}} = \sum_{|\beta|=k} \|D^\beta u\|_p$.

We know that I_1 is maximally non-compact (Hencl 03).

Note that L^{p^*} is not the optimal target space which is Lorentz space $L^{p^*,p}$. Consider now:

$$I_2 : V_0^{k,p}(\Omega) \to L^{p^*,q}(\Omega), \text{ with } p^* \leq q \leq \infty.$$

Then for $p* \leq q < \infty$ we have maximally non-compact embedding (Bouchala, 20). Question what about the target space $L^{p^*,\infty}$, i.e.

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Sobolev Embedding - case (iii)

$$I_3 : V_0^{k,p}(\Omega) \to L^{p^*,\infty}(\Omega).$$

Problem - $L^{p^*,\infty}(\Omega)$ is not disjointly superadditive.

Definition: We say that a (quasi)normed linear space $X(\Omega)$ containing functions defined on Ω is disjointly superadditive if there exist $\gamma > 0$ and $C > 0$ such that for every $m \in \mathbf{N}$ and every finite sequence of functions $\{f_k\}_{k=1}^m$ with pairwise disjoint supports in Ω one has

$$\sum_{k=1}^m \|f_k\|_{X(\Omega)}^\gamma \leq C \|\sum_{k=1}^m f_k\|_{X(\Omega)}^\gamma$$

Answer: I_3 is maximally non-compact embedding. (Musil, Olsak, Pick, L. 2020)

$$I_3 : V_0^{k,p}(\Omega) \to L^{p^*,q}(\Omega), \qquad p \leq q \leq \infty.$$

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Consider:

$$I_4 : V_0^k L^{n/k,1}(\Omega) \to L^\infty(\Omega), \qquad \Omega \subset \mathbf{R}^n, k \leq n$$

(the optimal target space L^∞!)
Using Triangle coloring problem we obtain:
$\beta(I) = 2^{-k/n} \|I_4\|$
Then I_4 is not maximally non-compact embedding.

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Let us consider:

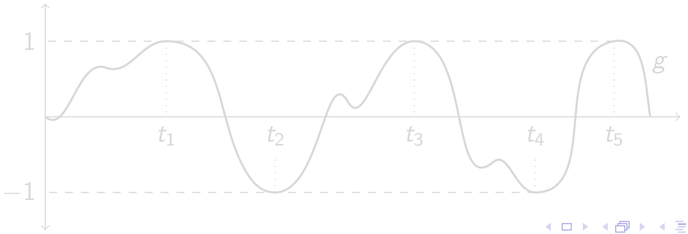
$$I_5 : V_0^1 L^{d,1}(Q) \to C(Q), \quad Q \text{ cube in } \mathbf{R}^d, d \geq 2.$$

and

$$I_6 : V_0^1 L^1(I) \to C(I), \quad I \subset \mathbf{R}$$

We need Zig-Zag theorem:
Let E be an n-dimensional subspace of $C(I)$ where I is any bounded closed interval. Then to every $\varepsilon > 0$ there exist a function $g \in E$, $\|g\|_\infty \leq 1 + \varepsilon$, and an n-tuple of points $t_1 < t_2 < \cdots < t_n$ in I such that

$$g(t_k) = (-1)^k \quad \text{for } 1 \leq k \leq n.$$



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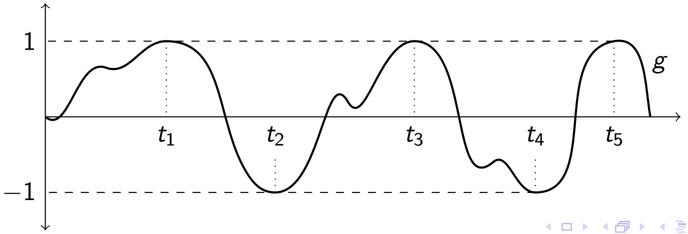

Sobolev Embedding - case (iii)

In case

$$I_6 : V_0^1 L^1(I) \to C(I), \quad I \subset \mathbf{R}$$

we have, use the above zig-zag theorem [L,Musil 18] and obtain:

$$s_n(I_6) = \frac{1}{2n}$$

where s_n stands for n-th Bernstein or isomorphism numbers,

$$s_n(I_6) = 1/2$$

where s_n stands for approximation or Gelfand numbers for every $n \geq 2$,

$$d_n(I_6) = 1/4$$

where d_n stands for n-th Kolmogorov number.

Strictly singular map

For embedding

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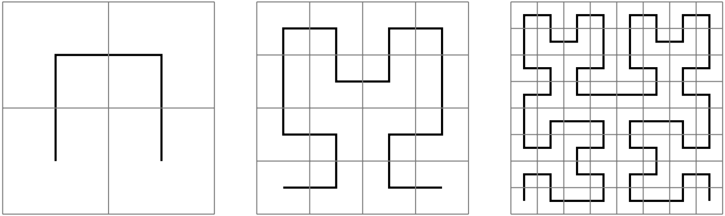

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We obtain [L,Musil 18]:

$$s_n(I_5) \asymp n^{-1/2}$$

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$$s_n(I_5) \asymp 1$$

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Generalization:
Let $X(Q)$ be any Banach function space over the cube Q in $\mathbf{R}^d$, $d \geq 2$, satisfying $X(Q) \subset L^{d,1}(\Omega)$. Then for every $n \in \mathbf{N}$

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In [Bourgain, Gromov 87] we have: Let $d \geq 1$ and Ω is the unit ball in $\mathbf{R}^d$. Then

$$b_n(I : W^{1,1}(\Omega) \to L_{d/(d-1)}(\Omega)) \leq c_d n^{-1/d}$$

where c_d only depends on d.

Natural Question: Are all extremal Sobolev embedding finitely strictly singular?

Answer: No (but in some cases yes)

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Let $\Omega \subseteq \mathbf{R}^d$ be a nonempty bounded open set, $m \in \mathbf{N}$, $1 \le m < d$, and $p \in [1, d/m)$.
Denote by I_p the identity operator $I_p \colon V_0^{m,p}(\Omega) \to L^{p^*,p}(\Omega)$, where $p^* = dp/(d-mp)$.
(i) We have

$$b_n(I) = \|I\| \qquad \text{for every } n \in \mathbf{N}, \tag{1}$$

where $\|I\|$ denotes the operator norm. Furthermore, I is not strictly singular.

(ii) Denote by I_{p^*} the identity operator $I_{p^*} \colon V_0^{m,p}(\Omega) \to L^{p^*}(\Omega)$, where $p^* = dp/(d-mp)$. There exists $n_0 \in \mathbf{N}$, depending only on d and m, such that

$$C_1 n^{-\frac{m}{d}} \le b_n(I_{p^*}) \le C_2 n^{-\frac{m}{d}} \qquad \text{for every } n \ge n_0. \tag{2}$$

Here C_1 and C_2 are constants depending only on d, m and p.
In particular, I_{p^*} is finitely strictly singular.

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