Ordinary differential operators and connections
Application to curvilinear webs

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Many thanks to the organizers of this conference for doing this job, despite of the war in their country. We all have a thought to them.

Thanks also for inviting me to participate.

Framework :
A priori C^∞ or holomorphic.
Analyticity (real or complex) required for thm 1.

Data :
- vector bundle $E \to V_n$ (rank p)
- vector bundle $F \to V_n$ (rank q)
- linear differential operator = epimorphism

$$D : J^k E \to F \to 0$$

[In practice, we only assume that D has a constant rank q at each point m of V, and replace the initial F by $Im(D)$].

Looking for solutions s of the PDE

$$(*) \quad \mathcal{D}s \equiv 0, \text{ where } \mathcal{D}s := D(j^k s).$$

$\mathcal{S}$:= vector space of solutions,
$\mathcal{S}_m$:= {germs of solutions at generic $m \in V$}.

Our aim :

1- Find an <u>upper-bound</u> for $dim(\mathcal{S}_m)$ (if any)

$$\pi(n, k, p, q).$$

2- Define a vector bundle $\mathcal{E}$ of rank $\pi(n, k, p, q)$ and a <u>connection</u> ∇ on it (when possible), s.t.

$$\mathcal{S} \cong \nabla - \text{invariant sections of } \mathcal{E}.$$

$$\Big(dim(\mathcal{S}_m = \pi(n, k, p, q)\Big) \Longleftrightarrow \Big(\text{the curvature vanishes}\Big).$$

3- Apply these results to abelian relations of <u>curvilinear webs</u> (definitions recalled below).

Notation : $c(n,h) := \binom{n-1+h}{h}$

- dimension of the vector space of homogeneous polynomials of degree h with n variables,
- rank of $S^hT^*(V)$,
- number of multi-indices $I=(i_1,...,i_n)$ of partial derivatives of order $|I|=h$,

Recall : $\sum_{\ell=0}^{h} c(n,\ell) = c(n+1,h)$.
(homogeneization of polynomials by adding one more variable)

Prolongations : $D_h : J^hE \to J^{h-k}F$ $(h \geq k)$ by all $(h-k)$ derivatives of $(*)$.

Formal solutions at order h :
$R_h := Ker(D_h)$
$R_{h+1} = J^1R_h \cap J^{h+1}E$ (intersection in J^1J^hE)

Method for aim nr.1 : Estimating the rank of formal solutions R_∞, and use analyticity.
First, estimating each step $R_h \to R_{h-1}$ of the tower

$$R_\infty \to ... \to R_h \to R_{h-1} \to ... \to R_k \to J^{k-1}E.$$

Linear systems $\Sigma(r_{h-1})$

Let $r_{h-1} \in R_{h-1}$.
$\Sigma(r_{h-1}) =$ linear system of equations. whose set $Sol(r_{h-1})$ of solutions is $\{r_h \in R_h$ over $r_{h-1}\}$.

In particular, for $r_{h-1} = 0$, the associated homogeneous system writes

$$< \sigma_h(D), r_h >= 0, \text{ where}$$

$\sigma_h(D)$, the **principal symbol** of D_h, denotes the restriction of D_h to $ker(J^hE \to J^{h-1}E) = S^hT^*(V) \otimes E$, (values into $S^{h-k}T^*(V) \otimes F$).

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & R_h^o & \to & R_h & \to & R_{h-1} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S^hT^*(V) \otimes E & \to & J^hE & \to & J^{h-1}E & \to & 0 \\
 & & \downarrow \sigma_h(D) & & \downarrow D_h & & \downarrow D_{h-1} & & \\
0 & \to & S^{h-k}T^*(V) \otimes F & \to & J^{h-k}F & \to & J^{h-k-1}F & \to & 0
\end{array}$$

nr.equations: $rk\Big(S^{h-k}T^*(V) \otimes F\Big) = q.c(n, h-k)$
nr.unknowns: $rk\Big(S^hT^*(V) \otimes E\Big) = p.c(n,h)$.

Ordinary differential operators

Definition 1 : *D is said to be <u>ordinary</u> if $\sigma_k(D)$ has ranq q, and all $\sigma_h(D)$ have max. rank*

$$inf\Big(c(n,h-k).q\ ,\ c(n,h).p\Big).$$

If $\frac{p}{q} < \frac{c(n,h-k)}{c(n,h)}$, then $\Sigma(r_{h-1})$ is overdetermined.

If $\frac{p}{q} \geq \frac{c(n,h-k)}{c(n,h)}$ (q not too big), $Sol(r_{h-1})$ is an affine space of dimension $c(n,h).p - c(n,h-k).q$.

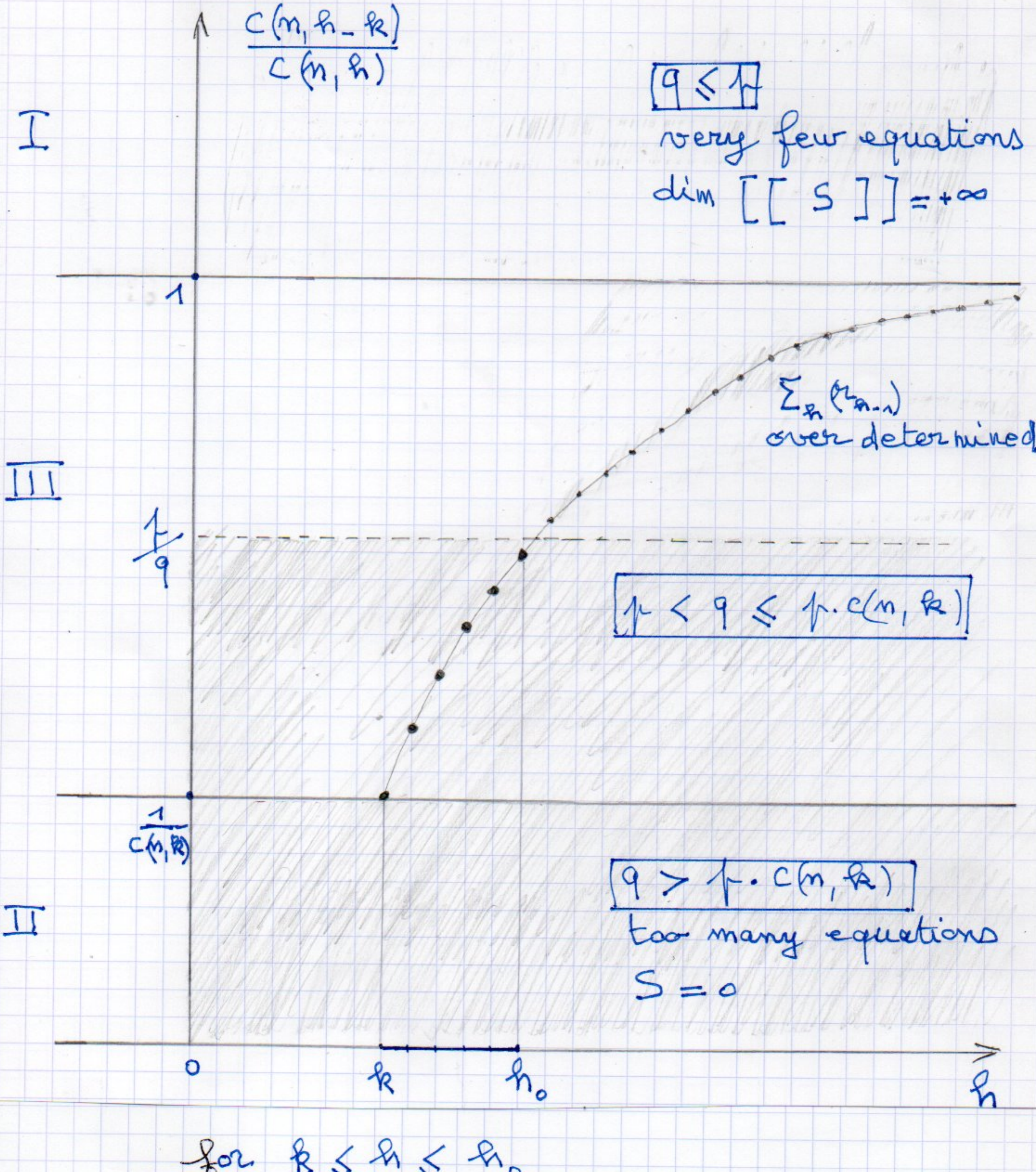
The function $\varphi : h \mapsto \frac{c(n,h-k)}{c(n,h)}$ is increasing from $\frac{1}{c(n,k)}$ to 1, when h goes from k to $+\infty$. Hence, 3 cases :

I : If $q \leq p$ (very few equations), then rank(R_h) increases to $+\infty$, and $dim(\mathcal{S})$ may be infinite.

II : If $q > p.c(n,k)$ (too many equations), then

$$dim(\mathcal{S}) = 0.$$

III : If $p < q \leq p.c(n,k)$, let $h_0 :=$ the biggest h ($h \geq k$) such that $q.c(n,h-k) \leq p.c(n,h)$ (the integral part of $\frac{p(n-1)}{q-p}$ when $k = 1$).



for $k \leq h \leq h_0$

$$p \cdot C(n, h) - q \cdot C(n, h - k) = C(n, h) \cdot \left(p - \mathcal{L}(h) \right).$$

case III : $p < q \leq p.c(n,k)$

<u>finite</u> number of conditions for D to be ordinary:

proposition 1 :
If $\sigma_h(D)$ has rank maximal for $h \leq h_0 + 1$ $\Big($for $h \leq h_0$ is sufficient, if $q.c(n, h_0 - k) = p.c(n, h_0)\Big)$, then D is ordinary.

behaviour of the sequence $\Big(\rho_h := rank(R_h)\Big)_h$
$\rho_k < \rho_{k+1} < ... < \rho_{h_0} \geq \rho_{h_0+1} \geq \rho_{h_0+2} \geq ... \geq \rho_\infty$

Proposition 2 : $0 \leq \rho_\infty \leq \rho_{h_0}$.

Theorem 1 :([L2]) *If the framework is <u>analytic</u> (real or complex), if $p < q \leq p.c(n,k)$, and if D is ordinary, then :*
- the space $\mathcal{S}_m$ of germs of solutions of $(*)$ at a point m of V is a finite dimensional vector space of dimension at most $\pi(k,n,p,q) := \rho_{h_0}$.

$$\rho_{h_0} = p.c(n+1, k-1) + \sum_{h=k}^{h_0} c(n,h).\Big(p - q.\varphi(h)\Big).$$

$$\Bigg(= p.c(n+1, h_0) - q.c(n+1, h_0 - k)\Bigg).$$

- a germ of solution is defined by its h_0-jet.

Proof : analyticity implies $dim(\mathcal{S}_m) \leq \rho_\infty$.

Connection in the ordinary calibrated case*

Definition 2 : *D is said to be calibrated if*

$$q.c(n, h_0 - k) = p.c(n, h_0),$$

(*if* $k = 1$, $h_0 = \frac{p(n-1)}{q-p}$, *which is an integer*).
Then, $R_{h_0} \xrightarrow{\pi_0} R_{h_0-1}$ is an isomorphism.

Recall : $R_{h_0} = \Big(J^1(R_{h_0-1}) \cap J^{h_0}E\Big) \subset J^1 J^{h_0-1}E)$.

$$u : R_{h_0-1} \overset{(\pi_0)^{-1}}{\longrightarrow} R_{h_0} \overset{\iota}{\hookrightarrow} J^1(R_{h_0-1})$$

$u := \iota \circ (\pi_0)^{-1}$ is a splitting of

$$0 \to T^*(V) \otimes R_{h_0-1} \to J^1 R_{h_0-1} \overset{u}{\underset{\longrightarrow}{\longleftarrow}} R_{h_0-1} \to 0,$$

$=$ connection on $\mathcal{E} := R_{h_0-1}$ $\Big(\text{rank } \pi(k, n, p, q)\Big)$,

$$\nabla \sigma = j^1 \sigma - u(\sigma).$$

Theorem 2 : ([L2])

(i) $\qquad \Big(s \in \mathcal{S}\Big) \Longleftrightarrow \Big(\nabla(j^{h_0-1}s) = 0\Big)$

(ii) $\qquad dim(\mathcal{S}) \leq dim(\mathcal{S}_m) \leq \pi(k, n, p, q)$

(iii) $\Big(dim(\mathcal{S}_m) = \pi(k, n, p, q)\Big) \Longleftrightarrow \Big(vanishing\ curvature\Big)$

*in the real case, does not need analyticity

Curvilinear webs

Data : d non-vanishing vector fields ∂_λ on V $(d > n)$, in general position, $\lambda = 1, ..., d$.

Abelian relations $\Big($of degree $(n-1)\Big) =$ family $(\eta_\lambda)_\lambda$ of $(n-1)$-forms on V s.t. $\sum_\lambda \eta_\lambda = 0$, and $\forall\lambda$, η_λ is ∂_λ-basic : $\iota_{\partial_\lambda}\eta_\lambda = 0, L_{\partial_\lambda}\eta_\lambda = 0$.

Definition of E :

$0 \to T_\lambda \to T(V) \to N_\lambda \to 0$

$0 \leftarrow T^*_\lambda \leftarrow T^*(V) \leftarrow N^*_\lambda \leftarrow 0$

$T_\lambda :=$ v.bundle generated by ∂_λ (rank 1),

$N_\lambda :=$ normal bundle (rank $n-1$),

$\bigwedge N^*_\lambda := \{$forms η on V s.t. $\iota_{\partial_\lambda}\eta = 0\}$,

$\bigoplus_{\lambda=1}^d \Big(\bigwedge^{n-1} N^*_\lambda\Big) \xrightarrow{Tr} \bigwedge^{n-1} T^*(V)$

$Tr\Big((\eta_\lambda)_{\lambda=1,\cdots,d}\Big) := \sum_{\lambda=1}^d \eta_\lambda$

$E := Ker(Tr)$, v.bundle (rank : $d-n$)

Definition of $\mathcal{D}$:

$L_\lambda := \iota_{\partial_\lambda} \circ d + d \circ \iota_{\partial_\lambda}$ (Lie derivative by ∂_λ),

$\mathcal{D}\Big((\eta_\lambda)_\lambda\Big) := \Big(L_\lambda(\eta_\lambda)\Big)_\lambda$.

Abelian relations $=$ solutions of $(\mathcal{D}s = 0)$.

Properties of $\mathcal{D}$

order $k=1$: $\mathcal{D}s$ depends only on the 1-jet j^1s,
rank E : $p=d-n$,
rank F : $q=d-1$,
case III : $p<q<p.n$,
D is always **ordinary**,
D is always **calibrated** : $\frac{p(n-1)}{q-p}$ is the integer

$$h_0=d-n,$$

$$\pi(n,1,d-n,d-1)=\sum_{h=0}^{d-n-1}\binom{n-2+h}{h}(d-n-h)$$

$$=\binom{d-1}{n},$$

recovering the Damiano's bound ([D]).

Theorem 3 : ([L])

The Damiano's bound for the rank of a curvilinear web is reached iff the curvature of the tautological connection on R_{d-n-1} vanishes.

Example of the $(n+3)$-web $W_{0,n+3}$:

$\partial_i = \frac{\partial}{\partial x_i}$ for any $i = 1, ..., n$
$\partial_{n+1} = \sum_i x_i \ \partial_i,$
$\partial_{n+2} = \sum_i (x_i - 1) \ \partial_i,$
$\partial_{n+3} = \sum_i x_i (x_i - 1) \ \partial_i.$

[The Bol's web for $n = 2$. More generally, $n + 2$ points in general position in the n-dimensional projective space, the pencils of straight lines through each of them, and the pencil of the rational algebraic curves of degree n going through each of them].

Damiano claimed : they all have maximal rank. He proved it for n even ([D]), and Pirio for n odd ([Pi]). Not easy : they needed to exhibit among them the so called "Euler relation", generalizing the relation of the dilogarithms in case $n = 2$.

For $n = 3$, we proved the vanishing of the curvature (using Maple), hence recovering the maximal rank 10, as well as maximality of the rank for the 4 and 5-subwebs, without exhibit any abelian relation .

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