Linearization, separability and Lax pair of the $a_4^{(2)}$ Toda lattice.

By : **LIETAP NDI BRUCE LIONNEL**

University of Maroua, Cameroon
Collaborator works Dehainsala Djagwa and Dongho Joseph

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Introduction

Soliton theory has an impact on pure mathematics. it has give an answer for the Schottky problem which, geometrically, consist to caracterise the jacobians amongst all the abelian varieties principaly polarised. Amongst the non linear equations which have soliton like solution, we have:
Kadomtsev-Petviashvili (KP), Sine Gordon, Boussinesq, Toda lattice.

The Toda lattice, introduced by Morikazu Toda in 1967, is a simple model for a one-dimensional crystal in solid-state physics. It is famous because it is one of the first examples of a completely integrable nonlinear system. It is described by a chain of particles with nearest-neighbor interaction, and its dynamics are governed by the Hamiltonian

$$H(p,q) = \sum_{n\in\mathbb{Z}} \left(\frac{p^2(n,t)}{2} + V\left(q(n+1,t) - q(n,t)\right) \right),$$

and the equations of motion

$$\begin{cases} \frac{d}{dt}p(n,t) = -\frac{\partial H(p,q)}{\partial q(n,t)} = e^{-(q(n,t)-q(n-1,t))} - e^{-(q(n+1,t)-q(n,t))} \\ \frac{d}{dt}q(n,t) = \frac{\partial H(p,q)}{\partial p(n,t)} = p(n,t) \end{cases}$$

where $q(n,t)$ is the displacement of the n-th particle from its equilibrium position, and $p(n,t)$ is its momentum (with mass $m = 1$).

We have two forms of Toda lattice, periodic and non periodic

- non periodic form where $q_0 = -\infty$ and $q_{n+1} = +\infty$

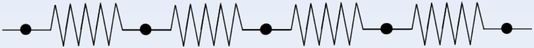

Figure 1-non periodic Toda Lattice

- periodic form where $q_{j+n} = q_j$ and $p_{j+n} = p_j$

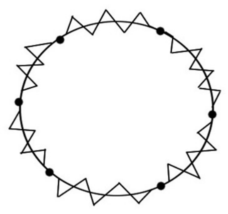

Figure 2- periodic Toda lattice.

The integrability of the periodic Toda lattice was established by Henon [8] and Flaschka [7] using the Lax pairs method. In 1976, Bogoyavlensky [4] introduced a generalization of the classical Toda periodic lattice to arbitrary Lie algebras.

Vanhaecke, Adler and Moerbeke in [3] give the explicit Hamiltonians for the periodic Toda lattices that involve precisely three (connected) particles. There are precisely six cases of them, going with the extended root systems $a_2^{(1)}$, $a_4^{(2)}$, $c_2^{(1)}$, $d_3^{(2)}$, $g_2^{(1)}$ and $d_4^{(3)}$.

The concept of algebraic complete integrability systems is a more general concept than complete integrability. An algebraic complete integrable system can be linearized on a complex torus, and its invariant functions (often called first integrals or constants) are polynomial maps and their restrictions to an invariant complex variety are meromorphic functions on a complex abelian variety. The fluxes generated by the constants of motion are straight lines in this complex abelian variety.

In [9], we have prove that the case $a_4^{(2)}$ of Toda lattice is an algebraic complete integrable system in the Adler-van Moerbeke sense and it is a two-dimensional integrable system. This system satisfies the linearization criterion [[2], theorem 6.41].

First, we prove that the generic fiber of the momentum map for this system is an affine part of an abelian surface. Second, we show that the flows of integrable vector fields on this surface are linear. Finally, using the formal Laurent solutions of the system, we provide a detailed geometric description of these abelian surfaces and the divisor at infinity.

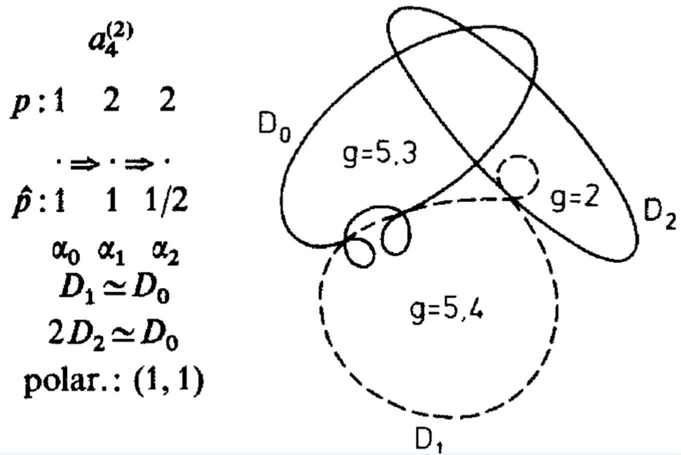

Figure 3- Curves completing the invariant surfaces $\mathbb{F}_c$ of the Toda lattice $a_4^{(2)}$ in abelian surfaces where $\mathcal{D}_i$ is the curve $\mathcal{D}_c^{(i)}$

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In the two-dimensional case, the invariant manifolds complete into Abelian surfaces by adding one (or several) curves to the affine surfaces. In this case, Vanhaecke proposed in [12] a method which leads to an explicit linearization of the vector field of the a.c.i. system. The computation of the first few terms of the Laurent solutions to the differential equations enables us to construct an embedding of the invariant manifolds in the projective space $\mathbb{P}^N$. From this embedding, one deduces the structure of the divisors $\mathcal{D}_c$ to be adjoined to the generic affine in order to complete them into Abelian surfaces $\mathbb{T}_c$. Thus, the system is a.c.i.. The different steps of the algorithm of Vanhaecke are given by:

case 1.

a) If one of the components of $\mathcal{D}_c$ is a smooth curve Γ_c of genus two, compute the image of the rational map $\phi_{[2\Gamma_c]} : \mathbb{T}_c^2 \to \mathbb{P}^3$ which is a singular surface in $\mathbb{P}^3$, the Kummer surface $\mathcal{K}_c$ of jacobian $Jac(\Gamma_c)$ of the curve Γ_c.

b) Otherwise, if one of the components of $\mathcal{D}_c$ is a $d:1$ unramified cover $\mathcal{C}_c$ of a smooth curve Γ_c of genus two, the map $p : \mathcal{C}_c \to \Gamma_c$ extends to the map $\widetilde{p} : \mathbb{T}_c^2 \to Jac(\Gamma_c)$. In this case, let $\mathcal{C}_c$ denote the (non complete) linear system $\widetilde{p}[2\Gamma_c] \subset [2\mathcal{C}_c]$ which corresponds to the complete linear system $[2\mathcal{C}_c]$ and compute now the Kummer surface $\mathcal{C}_c$ of $Jac(\Gamma_c)$ as image of $\phi_{\varepsilon_c} : \mathbb{T}_c^2 \to \mathbb{P}^3$.

c) Otherwise, change the divisor at infinity so as to arrive in case (a) or (b). This can always be done for any irreducible Abelian surface.

case 2. Choose a Weierstrass point W on the curve Γ_c and coordinates $(z_0 : z_1 : z_2 : z_3)$ for $\mathbb{P}^3$ such $\phi_{[2\Gamma_c]}(W) = (0 : 0 : 0 : 1)$ in case 1.(a) and $\phi_{\varepsilon_c}(W) = (0 : 0 : 0 : 1)$ in case 1.(b). Then this point will be a singular point (node) for the Kummer surface $\mathcal{K}_c$ whose equation is
$p_2(z_o; z_1; z_2)z_3^2 + p_3(z_o; z_1; z_2)z_3 + p_4(z_o; z_1; z_2) = 0$
where the p_i are polynomials of degree i. After a projective transformation which fixes $(0 : 0 : 0 : 1)$, we may assume that $p_2(z_o; z_1; z_2) = z_1^2 - 4z_0z_2$.
case 3. Finally, let s_1 and s_2 be the roots of the quadractic equation $z_0s^2 + z_1s + z_2 = 0$, whose discriminant is $p^2(z_o; z_1; z_2)$, with the z_i expressed in terms of the original variables. Then the differential equations describing the vector field of the system are rewritten by direct computation in the classical Weierstrass form

The main question it is :

how we can linearize and find the Toda lattice $a_4^{(2)}$ Lax pair or Lax representation?

General objectif

Find the Toda lattice $a_4^{(2)}$ Lax pair or Lax representation?

Specifics objectifs

- Linearise our curve of system which has a genus two;
- construct an explicit map from the generic fiber $\mathbb{F}_c$ into the Jacobian of the Riemann surface $\overline{\Gamma}_c$. After we find the kummer surface of $Jac(\mathcal{K}_c)$
- find $u(\lambda)$,$v(\lambda)$ and $f(\lambda)$. According to Mumford's description of hyperelliptic Jacobians (see [[10], Section 3.1]), if Γ is a hyperelliptic curve of genus two then the Riemann surface $\overline{\Gamma}$ is embedded in its jacobian such that $Jac(\overline{\Gamma})$ Γ is isomorphic to the space of pairs of polynomials $(u(\lambda); v(\lambda))$. $u(\lambda)$ is a monic of degree two and $v(\lambda)$ less than two. $f(\lambda) - v^2(\lambda)$ is divisible by $u(\lambda)$.

results presentations.

Proposition 4.1

[9] For the Laurent solution $x\left(t;m_2\right)$ restricted to the invariant surface, for $c\in\Omega$, the Painlevé divisor $\Gamma_c^{(2)}$ is a smooth two genus hyperelliptic curve. It is given by

$$\Gamma_c^{(2)}: e^4a^4-\left(8c_1+c_2e^2\right)a^2e^2-64e^5+4e^2c_1c_2+4c_3e^4+16c_1^2=0.$$

It is completed in a Riemann surface, which is a double covering of $\mathbb{P}^1$ ramified into 5 points .

In [9], we have prove the above proposition. Notice that $\Gamma_c^{(2)}$ can be writted by $X=\frac{1}{256}\left[\left(2t-c_2\right)^2+16c_3-c_2^2\right]=0$ with $X=e$ and $t=a^2-\frac{4c_1}{e^2}$.

results presentations.

Proposition 4.2

Let $\mathcal{K}_c$ *be the quotient of the curve* $\overline{\Gamma}_c^{(2)}$ *by the involution* σ *(defined on* $\mathbb{C}^6$ *by*

$$\sigma\left(x_0,x_1,x_2,y_0,y_1,y_2\right)=\left(x_0,x_1,x_2,-y_0,-y_1,-y_2\right)$$

preserves the constants of motion F_1,F_2 *and* F_3*, define in our paper [9], hence leave the fibers of the momentum map* F *invariant). For generic* c*, the quotient curve* $\mathcal{K}_c$ *is a smooth curve of genus two and the map* $\overline{\Gamma}_c^{(2)} \to \mathcal{K}_c$ *is an unramified* $2:1$ *map.*

The curve $\mathcal{K}_c : Z^2 = t\left(\frac{1}{256}\left[(2t-c_2)^2 - c_2^2 + 16c_3\right]\right)^2 + 4c_1$

results presentations.

The functions

$$\theta_0 = 1 \ , \ \theta_1 = x_2 \ , \ \theta_2 = x_1 x_2 + 4x_2^2 - y_2^2 x_2 \text{ and } \theta_3 = x_1 x_2^2, \quad (2)$$

allow us to embed the Kummer surface of $Jac(\mathcal{K}_c)$ in the projective space $\mathcal{P}^3$. Consider now the Koidara map which correspond to these functions.

$$\begin{array}{llcl} \varphi_c : & Jac(\overline{\Gamma}_c) & \longrightarrow & \mathcal{P}^3 \\ & p = (x_0, x_1, x_2, y_0, y_2) & \longmapsto & (\theta_0(p) : \theta_1(p) : \theta_2(p) : \theta_3(p)) \end{array}$$

Since the functions θ_i correspond to the sections of the line bundle $[2D_c^2]$, the map φ_c maps the $Jac(\mathcal{K}_c)$ into its Kummer surface, which is a singular quartic in the projective space $\mathcal{P}^3$.

results presentations.

computed by eliminating the variables $x_0; x_1; x_2; y_0; y_2$ from (2), the result is a quartic equation of the Kummer surface of $Jac(\mathcal{K}_c)$ which it can be put in the following form :

$$\left((c_2 + 16\theta_1)^2 - 16\,(16\theta_2 + 4c_2\theta_1 + c_3)\right)\theta_3^2 + 2\theta_3 P_3\,(\theta_1,\theta_2) + P_4\,(\theta_1,\theta_2) = 0 \tag{3}$$

where P_3 is a polynom of degree 3 and P_4 a polynom of degree 4 in θ_1 and θ_2 given by

$$\begin{aligned} P_3\,(\theta_1,\theta_2) &= -\,(c_2 + 16\theta_1)\left(\theta_2\,(\theta_1 c_2 + 4\theta_2) + c_3\theta_1^2\right) - 64c_1 \\ P_4\,(\theta_1,\theta_2) &= \left(c_3\theta_1^2 + 4\theta_2^2\right)^2 - \theta_1\left(-2\theta_1^2 c_2\theta_2 c_3 + 256c_1\theta_1^2 - \theta_1 c_2^2\theta_2^2\right. \\ &= \left. -64c_1\theta_2 - 8\theta_2^3 c_2\right) \end{aligned}$$

results presentations.

Proposition 4.3

Using the coefficient of θ_3^2 in (3) and according to Vanhaecke [[12], Theorem 9], the vector field $\mathcal{V}$ define the Toda lattice linearizes upon the setting

$$\begin{aligned} u(\lambda) &= \lambda^2 + (c_2 + 16x_2)\lambda + 4x_2\left(-16y_2^2 + 64x_2 + 16x_1 + 4c_2\right) + 4c_3 \\ &= \lambda^2 + \left(y_0^2 + 4y_2^2 - 4x_0 - 8x_1\right)\lambda + (4x_1 - 2y_0y_2)^2 - 16x_0y_2^2. \end{aligned}$$

and the roots of polynom λ_1 and λ_2 verify

$$\lambda_1 + \lambda_2 = -16x_2 - c_2 \text{ and } \lambda_1\lambda_2 = 4x_2\left(-16y_2^2 + 64x_2 + 16x_1 + 4c_2\right) + 4c_3 \quad (4)$$

with respect to the vector field $\mathcal{V}_1$, define in [9],

$$\dot{\lambda}_1 + \dot{\lambda}_2 = -16x_2y_2 \text{ and } \dot{\lambda}_1\lambda_2 + \lambda_1\dot{\lambda}_2 = -16x_2\left(-y_2\left(y_0^2 - 4x_0\right) + 2x_1y_0\right)$$

results presentations.

Substituting (4) and (5) in the invariants and eliminating variables $x_0; x_1; x_2; y_0; y_2$, we obtain two quadratics polynomials

$$\dot{\lambda}_i^2 = \frac{\lambda_i^5 + 2c_2\lambda_i^4 + \left(8c_3 + c_2^2\right)\lambda_i^3 + 8c_2c_3\lambda_i^2 + 16c_3^2\lambda_i - 16384c_1}{4\left(\lambda_1 - \lambda_2\right)^2}, \; i = 1,2$$

which verify the mumford system

$$\frac{\dot{\lambda_1}}{\sqrt{f(\lambda_1)}} + \frac{\dot{\lambda_2}}{\sqrt{f(\lambda_2)}} = 0 \text{ and } \frac{\lambda_1\dot{\lambda_1}}{\sqrt{f(\lambda_1)}} + \frac{\lambda_2\dot{\lambda_2}}{\sqrt{f(\lambda_2)}} = \frac{1}{2i}dt \qquad (6)$$

with

$$f(\lambda) = \lambda_i^5 + 2c_2\lambda_i^4 + \left(8c_3 + c_2^2\right)\lambda_i^3 + 8c_2c_3\lambda_i^2 + 16c_3^2\lambda_i - 16384c_1$$

It follows that the Toda system is linearized on the Jacobian variety of

results presentations.

Théorème 4.1

The Lax equation for the Hamiltonian vector field $\mathcal{V}_1$ is given by

$$\dot{X}(\lambda) = [X(\lambda), Y(\lambda)]$$

with

$$X(\lambda) = \begin{pmatrix} v(\lambda) & u(\lambda) \\ w(\lambda) & -v(\lambda) \end{pmatrix} \text{ and } Y(\lambda) = \begin{pmatrix} 0 & 1 \\ b(\lambda) & 0 \end{pmatrix}$$

where $u(\lambda)$, $v(\lambda)$ $w(\lambda)$ and $b(\lambda)$ are defined by

$$(x_0, x_1, x_2, y_0, y_2) \longmapsto \begin{cases} u(\lambda) = & \lambda^2 + u_1\lambda + u_0 \\ v(\lambda) = & v_1\lambda + v_0 \\ w(\lambda) = & \lambda^3 + w_2\lambda^2 + w_1\lambda + w_0 \end{cases} \quad (7)$$

results presentations.

with

$$u_0 = (4x_1 - 2y_0y_2)^2 - 16x_0y_2^2 \qquad v_0 = 16x_2\left(y_2\left(y_0^2 - 4x_0\right) - 2x_1y_0\right)$$
$$u_1 = -\left(y_0^2 + 4y_2^2 - 4x_0 - 8x_1\right) \qquad v_1 = 16x_2y_2$$

$$\begin{aligned} w_0 =\ & 256y_0^2x_2^2 - 1024x_0x_2^2 - 768x_2^2y_2^2 \\ w_1 =\ & 16x_1^2 + 4y_0^2y_2^2 - 32y_0^2x_2 - 16x_0y_2^2 + 128x_0x_2 + 256x_2^2 \\ & +128x_2x_1 - 16x_1y_0y_2 \\ w_2 =\ & y_0^2 - 8x_1 - 32x_2 - 4x_0 + 4y_2^2 \end{aligned}$$

and the coefficient $b(\lambda)$ of the matrix $Y(\lambda)$ is the polynomial part of the rational function $\frac{w(\lambda)}{u(\lambda)}$

Thanks for ATTENTION !!!

- Adler, M., Moerbeke, P.V, *La géométrie complexe de l'analyse de Kowalewski-Painlevé*, Invent. Math. 97 (1989) 3-51.
- Adler, M., Moerbeke,P.V and Vanhaecke, P., *Intégrabilité algébrique, géométrie Painlevé et algèbres de Lie*, Ergeb. Math. Grenzgeb. 47 (3) (2004) Berlin- Heidelberg : Springer.
- Adler, M. and Moerbeke, P.V, *The Toda Lattice, Dynkin Diagrams, Singularities and Abelian Varieties*, Invent. Math., 1991, vol. 103, no. 2, pp. 223-278.
- Bogoyavlensky, O. I., *On Perturbations of the Periodic Toda Lattice*, Comm. Math. Phys., 1976, vol. 51, vol. 3, pp. 201-209.

- Dehainsala, D., *Sur l'intégrabilité Algébrique Des Réseaux De Toda : Cas Particuliers Des Réseaux $c_2^{(1)}$ et $d_3^{(2)}$*, (Thèse de doctorat), Université de Poitiers, 2008.
- Dehainsala, D., *Algebraic complete integrability of the $c_2^{(1)}$ Toda lattice*, University of NDjamena, Journal of Geometry and Physics 135 (2019) 80-97.
- Flaschka, H., *The Toda Lattice: 1. Existence of Integrals*, Phys. Rev. B (3), 1974, vol. 9, pp. 1924-1925.
- Hénon, M., *Integrals of the Toda Lattice*, Phys. Rev. B(3), 1974, vol. 9, pp. 1921-1923.
- Lietap, N.,Dehainsala, D. and Dongho, J., *Algebraic complete integrability of the $a_4^{(2)}$ Toda lattice*, University

of maroua,https://doi.org/10.48550/arXiv.2404.13688, preprint paper.

- Mumford, D.: Tata Lectures on Theta: 2. Jacobian Theta Functions and Differential Equations, Progr. Math., vol. 43, Boston, MA: Birkhäuser, 1984.

- Toda, M., *One-Dimensional Dual Transformation*, J. Phys. Soc. Japan, 1965, vol. 20, no. 11, pp. 2095A (see also Progr. Theoret. Phys. Suppl., 1966, no. 36, pp. 113-119).

- Vanhaecke, P., Linearising two-dimensional integrable systems and the construction of action-angle variables. Math. Z., 1992, vol. 211, no. 2, pp. 265-313.