Vector bundle construction via monads on multiprojective Spaces

Damian Maingi
International scientific online conference
Algebraic and geometric methods of analysis 2024

May 27, 2024

Outline

Introduction

In this presentation I aim to construct monads on multiprojective spaces and study their vector bundles.

Introduction

In this presentation I aim to construct monads on multiprojective spaces and study their vector bundles.
We shall denote by X the space over which to construct our monad and the vector bundle associated to the monad we denote by E.

In this presentation I aim to construct monads on multiprojective spaces and study their vector bundles.

We shall denote by X the space over which to construct our monad and the vector bundle associated to the monad we denote by E.

I will first set out to establish the existence of monads on a multiprojective space $\mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$ and then on $\mathbf{P}^{2n+1} \times \cdots \times \mathbf{P}^{2n+1}$.

Introduction

In this presentation I aim to construct monads on multiprojective spaces and study their vector bundles.
We shall denote by X the space over which to construct our monad and the vector bundle associated to the monad we denote by E.
I will first set out to establish the existence of monads on a multiprojective space $\mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$ and then on $\mathbf{P}^{2n+1} \times \cdots \times \mathbf{P}^{2n+1}$.
We will not distinguish between a vector bundle E on X and its locally free sheaf of section. Why?

Lemma

The functor which associates the locally free sheaf $\mathscr{E} = \mathcal{O}_X(E)$ to a vector bundle E on X is an equivalence of categories between the category of vector bundles of rank r over X and the category of locally free sheaves of rank r on X.

Motivation

First observe that the difficulty in constructing non-splitting vector bundles on algebraic varieties increases when the difference between the rank of the bundle and the dimension of the variety increases.

Motivation

First observe that the difficulty in constructing non-splitting vector bundles on algebraic varieties increases when the difference between the rank of the bundle and the dimension of the variety increases.
Indeed, the most interesting problem is to find indecomposable vector bundles of low rank comparing with the dimension of the ambient space.

Motivation

First observe that the difficulty in constructing non-splitting vector bundles on algebraic varieties increases when the difference between the rank of the bundle and the dimension of the variety increases.
Indeed, the most interesting problem is to find indecomposable vector bundles of low rank comparing with the dimension of the ambient space.
In this context we have the famous Hartshorne's conjecture concerning the non-existence of indecomposable rank 2 vector bundles on n-dimensional projective spaces for $n \geq 7$.

Background

Monads appear in many contexts within algebraic geometry and they are very useful in construction of vector bundles with prescribed invariants like rank, determinants, chern class etc.

Background

Monads appear in many contexts within algebraic geometry and they are very useful in construction of vector bundles with prescribed invariants like rank, determinants, chern class etc.
They were first introduced by Horrocks who showed that all vector bundles E on $\mathbf{P}^3$ could be obtained as the cohomology bundle of a monad of the following kind:

Background

Monads appear in many contexts within algebraic geometry and they are very useful in construction of vector bundles with prescribed invariants like rank, determinants, chern class etc.
They were first introduced by Horrocks who showed that all vector bundles E on $\mathbf{P}^3$ could be obtained as the cohomology bundle of a monad of the following kind:

$$0 \longrightarrow \oplus_i \mathcal{O}_{\mathbf{P}^3}(a_i) \xrightarrow{A} \oplus_j \mathcal{O}_{\mathbf{P}^3}(b_j) \xrightarrow{B} \oplus_n \mathcal{O}_{\mathbf{P}^3}(c_n) \longrightarrow 0$$

where A and B are matrices whose entries are homogeneous polynomials of degrees $b_j - a_i$ and $c_n - b_j$ respectively for some integers i, j, n.

Monads

Definition

Let X be a smooth projective variety. A *monad* on X is a complex of vector bundles:$M_\bullet$ $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ with α injective and β surjective.

The sheaf $E = ker(\beta)/im(\alpha)$ is called the cohomology sheaf of the monad $M_\bullet$

Display of the monad:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & K & \longrightarrow & E & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow[\alpha]{} & B & \longrightarrow & Q & \longrightarrow & 0 \\
 & & & & {\scriptstyle\beta}\downarrow & & \downarrow & & \\
 & & & & C & = & C & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

From which we have:
(i) $K = kernel(\beta)$
(ii)$Q = cokernel(\beta)$
(iii)$E = ker(\beta)/im(\alpha)$
(iv) rank(E) = rank(B) - rank(A) - rank(C)
(v) $c_t(E) = c_t(B) \cdot c_t(A)^{-1} \cdot c_t(C)^{-1}$ and particularly
$c_1(E) = c_1(B) - c_1(A) - c_1(C)$.

More definitions

Definition

Let X be a nonsingular projective variety, let $\mathscr{L}$ be a very ample invertible sheaf, and V, W, U be finite dimensional k-vector spaces. A linear monad on X is a short complex of sheaves,

$$0 \longrightarrow V \otimes \mathscr{L}^{-1} \xrightarrow{\alpha} W \otimes \mathcal{O}_X \xrightarrow{\beta} U \otimes \mathscr{L} \longrightarrow 0$$

where $\alpha \in Hom(V, W) \otimes H^0\mathscr{L}$ is injective and $\beta \in Hom(W, U) \otimes H^0\mathscr{L}$ is surjective.

More definitions

Definition

Let X be a nonsingular projective variety, let $\mathscr{L}$ be a very ample invertible sheaf, and V, W, U be finite dimensional k-vector spaces. A linear monad on X is a short complex of sheaves,

$$0 \longrightarrow V \otimes \mathscr{L}^{-1} \xrightarrow{\alpha} W \otimes \mathcal{O}_X \xrightarrow{\beta} U \otimes \mathscr{L} \longrightarrow 0$$

where $\alpha \in Hom(V, W) \otimes H^0\mathscr{L}$ is injective and $\beta \in Hom(W, U) \otimes H^0\mathscr{L}$ is surjective.

Definition

A torsion free sheaf E on X is said to be a *linear sheaf* on X if it can be represented as the cohomology sheaf of a linear monad i.e. $E = ker(\beta)/im(\alpha)$, moreover $rank(E) = w - u - v$, where $w = \dim W$, $v = \dim V$ and $u = \dim U$.

Existence of Monads

Fløystad, [Communications in Algebra, 28 (2000)]

Lemma

Let $k \geq 1$. There exists monads on $\mathbf{P}^k$ whose maps are matrices of linear forms,

$$0 \longrightarrow \mathcal{O}^a_{\mathbf{P}^k}(-1) \xrightarrow{\alpha} \mathcal{O}^b_{\mathbf{P}^k} \xrightarrow{\beta} \mathcal{O}^c_{\mathbf{P}^k}(1) \longrightarrow 0$$

if and only if at least one of the following is fulfilled;
(1) $b \geq 2c + k - 1$, $b \geq a + c$ *and*
(2) $b \geq a + c + k$

Theorem on existence

D Maingi [Le Matematiche. Vol. LXIX (2014)]

Theorem

Let $X = \mathbf{P}^n \times \mathbf{P}^m$ *and let* $\mathscr{L} = \mathcal{O}_X(\rho, \sigma)$ *be an ample line bundle on* X*. Denote by* $N = h^0(\mathcal{O}_X(\rho, \sigma)) - 1$*. Let* α, β, γ *be positive integers such that at least one of the following conditions holds*
(1) $\beta \geq 2\gamma + N - 1$*, and* $\beta \geq \alpha + \gamma$*,*
(2) $\beta \geq \alpha + \gamma + N$*.*
Then, there exists a linear monad on X *of the form*

$$0 \longrightarrow \mathcal{O}_X^{\alpha}(-\rho, -\sigma) \xrightarrow{A} \mathcal{O}_X^{\beta} \xrightarrow{B} \mathcal{O}_X^{\gamma}(\rho, \sigma) \longrightarrow 0$$

Theorem on existence

D Maingi [Open Journal of Mathematical Sciences, OMS - Vol 7 (2023)]

Theorem

Let n, m and k be positive integers. Then there exists a linear monad on $X = \mathbf{P}^n \times \mathbf{P}^n \times \mathbf{P}^m \times \mathbf{P}^m$ of the form;

$$0 \longrightarrow \mathcal{O}_X(-1,-1,-1,-1)^{\oplus k} \xrightarrow[f]{} \mathscr{G}_n \oplus \mathscr{G}_m \xrightarrow[g]{} \mathcal{O}_X(1,1,1,1)^{\oplus k}$$

where $\mathscr{G}_n := \mathcal{O}_X(0,-1,0,0)^{\oplus n+k} \oplus \mathcal{O}_X(-1,0,0,0)^{\oplus n+k}$ *and*
$\mathscr{G}_m := \mathcal{O}_X(0,0,-1,0)^{\oplus m+k} \oplus \mathcal{O}_X(0,0,0,-1)^{\oplus m+k}$.

Theorem on existence

D Maingi [Manuscripta Mathematica (2023)]

Theorem

Let $a_1, \cdots, a_n$ and k be positive integers. Then there exists a linear monad on $X = \mathbf{P}^{a_1} \times \mathbf{P}^{a_1} \times \mathbf{P}^{a_2} \times \mathbf{P}^{a_2} \times \cdots \times \mathbf{P}^{a_n} \times \mathbf{P}^{a_n}$ of the form;

$$0 \to \mathcal{O}_X(-1, \cdots, -1)^{\oplus k} \xrightarrow{f} \mathscr{G}_1 \oplus \cdots \oplus \mathscr{G}_n \xrightarrow{g} \mathcal{O}_X(1, \cdots, 1)^{\oplus k} \to 0$$

where

$\mathscr{G}_1 := \mathcal{O}_X(-1, 0, 0, \cdots, 0)^{\oplus a_1 + \oplus k} \oplus \mathcal{O}_X(0, -1, 0, 0, \cdots, 0)^{\oplus a_1 + \oplus k}$
$\mathscr{G}_2 := \mathcal{O}_X(0, 0, -1, \cdots, 0)^{\oplus a_2 + \oplus k} \oplus \mathcal{O}_X(0, 0, 0, -1, \cdots, 0)^{\oplus a_2 + \oplus k}$
$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$
$\mathscr{G}_n := \mathcal{O}_X(0, 0, \cdots, 0, -1, 0)^{\oplus a_n + \oplus k} \oplus \mathcal{O}_X(0, 0, \cdots, 0, -1)^{\oplus a_n + \oplus k}$

Theorem on existence

D Maingi [GOAL]

Theorem

Let $X = \mathbf{P}^{a_1} \cdots \times \mathbf{P}^{a_n}$ *and* $\mathscr{L} = \mathcal{O}_X(\alpha_1, \cdots, \alpha_t)$ *an ample line bundle. Denote by* $N = h^0(\mathcal{O}_X(\alpha_1, \cdots, \alpha_t)) - 1$*. Then there exists a linear monad* $M_\bullet$ *on* X *of the form*

$$0 \to \mathcal{O}_X(-\alpha_1, \cdots, -\alpha_t)^{\oplus \alpha} \xrightarrow[f]{} \mathcal{O}_X^{\oplus \beta} \xrightarrow[g]{} \mathcal{O}_X(\alpha_1, \cdots, \alpha_t)^{\oplus \gamma} \to 0$$

if atleast one of the following is satified

a. $\beta \geq 2\gamma + N - 1,$ *and* $\beta \geq \alpha + \gamma,$
b. $\beta \geq \alpha + \gamma + N,$ *where* α, β, γ *be positive integers.*

Proof.

For the ample line bundle $\mathscr{L} = \mathcal{O}_X(\alpha_1, \ldots, \alpha_t)$ we have the Segre embedding

$$i^* : X = \mathbf{P}^{a_1} \cdots \times \mathbf{P}^{a_n} \hookrightarrow \mathbf{P}\big(H^0(X, \mathcal{O}_X(\alpha_1, \ldots, \alpha_t))\big) \cong \mathbf{P}^N$$

where $N = \left(\binom{a_1+\alpha_1}{\alpha_1}\binom{a_2+\alpha_2}{\alpha_2}\cdots\binom{a_n+\alpha_t}{\alpha_t}\right) - 1.$

Suppose that one of the conditions of Fløystad's is satified thus there exists a linear monad

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^N}(-1)^{\oplus\alpha} \xrightarrow[A]{} \mathcal{O}_{\mathbf{P}^N}^{\oplus\beta} \xrightarrow[B]{} \mathcal{O}_{\mathbf{P}^N}(1)^{\oplus\gamma} \longrightarrow 0$$

on $\mathbf{P}^N$ whose morphisms are matrices A and B with entries monomials of degree one where □

Proof.

For the ample line bundle $\mathscr{L} = \mathcal{O}_X(\alpha_1, \ldots, \alpha_t)$ we have the Segre embedding

$$i^* : X = \mathbf{P}^{a_1} \cdots \times \mathbf{P}^{a_n} \hookrightarrow \mathbf{P}\big(H^0(X, \mathcal{O}_X(\alpha_1, \ldots, \alpha_t))\big) \cong \mathbf{P}^N$$

where $N = \left(\binom{a_1+\alpha_1}{\alpha_1}\binom{a_2+\alpha_2}{\alpha_2}\cdots\binom{a_n+\alpha_t}{\alpha_t}\right) - 1.$

Suppose that one of the conditions of Fløystad's is satified thus there exists a linear monad

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^N}(-1)^{\oplus\alpha} \xrightarrow[A]{} \mathcal{O}_{\mathbf{P}^N}^{\oplus\beta} \xrightarrow[B]{} \mathcal{O}_{\mathbf{P}^N}(1)^{\oplus\gamma} \longrightarrow 0$$

on $\mathbf{P}^N$ whose morphisms are matrices A and B with entries monomials of degree one where □

Proof.

For the ample line bundle $\mathscr{L} = \mathcal{O}_X(\alpha_1, \ldots, \alpha_t)$ we have the Segre embedding

$$i^* : X = \mathbf{P}^{a_1} \cdots \times \mathbf{P}^{a_n} \hookrightarrow \mathbf{P}\big(H^0(X, \mathcal{O}_X(\alpha_1, \ldots, \alpha_t))\big) \cong \mathbf{P}^N$$

where $N = \left(\binom{a_1+\alpha_1}{\alpha_1}\binom{a_2+\alpha_2}{\alpha_2}\cdots\binom{a_n+\alpha_t}{\alpha_t}\right) - 1.$
Suppose that one of the conditions of Fløystad's is satified thus there exists a linear monad

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^N}(-1)^{\oplus\alpha} \xrightarrow[A]{} \mathcal{O}_{\mathbf{P}^N}^{\oplus\beta} \xrightarrow[B]{} \mathcal{O}_{\mathbf{P}^N}(1)^{\oplus\gamma} \longrightarrow 0$$

on $\mathbf{P}^N$ whose morphisms are matrices A and B with entries monomials of degree one where □

Proof.

For the ample line bundle $\mathscr{L} = \mathcal{O}_X(\alpha_1, \ldots, \alpha_t)$ we have the Segre embedding

$$i^* : X = \mathbf{P}^{a_1} \cdots \times \mathbf{P}^{a_n} \hookrightarrow \mathbf{P}\big(H^0(X, \mathcal{O}_X(\alpha_1, \ldots, \alpha_t))\big) \cong \mathbf{P}^N$$

where $N = \left(\binom{a_1+\alpha_1}{\alpha_1}\binom{a_2+\alpha_2}{\alpha_2}\cdots\binom{a_n+\alpha_t}{\alpha_t}\right) - 1.$
Suppose that one of the conditions of Fløystad's is satified thus there exists a linear monad

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^N}(-1)^{\oplus\alpha} \xrightarrow[A]{} \mathcal{O}_{\mathbf{P}^N}^{\oplus\beta} \xrightarrow[B]{} \mathcal{O}_{\mathbf{P}^N}(1)^{\oplus\gamma} \longrightarrow 0$$

on $\mathbf{P}^N$ whose morphisms are matrices A and B with entries monomials of degree one where □

Proof.

$A \in (\mathcal{O}_{\mathbf{P}^N}(-1)^{\oplus\alpha}, \mathcal{O}_{\mathbf{P}^N}^{\oplus\beta}) \cong H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(1)^{\oplus\alpha\beta})$
$B \in (\mathcal{O}_{\mathbf{P}^N}^{\oplus\beta}, \mathcal{O}_{\mathbf{P}^N}(1)^{\oplus\gamma}) \cong H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(1)^{\oplus\beta\gamma})$.
Thus, A and B induce a monad on X,

$$0 \longrightarrow \mathscr{L}^{-1\oplus\alpha} \xrightarrow{\bar{A}} \mathcal{O}_X^{\oplus\beta} \xrightarrow{\bar{B}} \mathscr{L}^{\oplus\gamma} \longrightarrow 0$$

where whose morphisms are matrices $\bar{A}$ and $\bar{B}$ with entries multidegree monomials such that $\bar{A} \in (\mathcal{O}_X(-\alpha_1, \ldots, -\alpha_t)^{\oplus\alpha}, \mathcal{O}_X^{\oplus\beta})$ and $\bar{B} \in (\mathcal{O}_X^{\oplus\beta}, \mathcal{O}_X(\alpha_1, \ldots, \alpha_t)^{\oplus\gamma})$ □

Proof.

$A \in (\mathcal{O}_{\mathbf{P}^N}(-1)^{\oplus\alpha}, \mathcal{O}_{\mathbf{P}^N}^{\oplus\beta}) \cong H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(1)^{\oplus\alpha\beta})$
$B \in (\mathcal{O}_{\mathbf{P}^N}^{\oplus\beta}, \mathcal{O}_{\mathbf{P}^N}(1)^{\oplus\gamma}) \cong H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(1)^{\oplus\beta\gamma})$.
Thus, A and B induce a monad on X,

$$0 \longrightarrow \mathscr{L}^{-1\oplus\alpha} \xrightarrow{\bar{A}} \mathcal{O}_X^{\oplus\beta} \xrightarrow{\bar{B}} \mathscr{L}^{\oplus\gamma} \longrightarrow 0$$

where whose morphisms are matrices $\bar{A}$ and $\bar{B}$ with entries multidegree monomials such that $\bar{A} \in (\mathcal{O}_X(-\alpha_1, \ldots, -\alpha_t)^{\oplus\alpha}, \mathcal{O}_X^{\oplus\beta})$ and $\bar{B} \in (\mathcal{O}_X^{\oplus\beta}, \mathcal{O}_X(\alpha_1, \ldots, \alpha_t)^{\oplus\gamma})$ □

Proof.

$A \in (\mathcal{O}_{\mathbf{P}^N}(-1)^{\oplus\alpha}, \mathcal{O}_{\mathbf{P}^N}^{\oplus\beta}) \cong H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(1)^{\oplus\alpha\beta})$
$B \in (\mathcal{O}_{\mathbf{P}^N}^{\oplus\beta}, \mathcal{O}_{\mathbf{P}^N}(1)^{\oplus\gamma}) \cong H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(1)^{\oplus\beta\gamma})$.
Thus, A and B induce a monad on X,

$$0 \longrightarrow \mathscr{L}^{-1^{\oplus\alpha}} \xrightarrow{\bar{A}} \mathcal{O}_X^{\oplus\beta} \xrightarrow{\bar{B}} \mathscr{L}^{\oplus\gamma} \longrightarrow 0$$

whose morphisms are matrices $\bar{A}$ and $\bar{B}$ with entries multidegree monomials such that $\bar{A} \in (\mathcal{O}_X(-\alpha_1, \ldots, -\alpha_t)^{\oplus\alpha}, \mathcal{O}_X^{\oplus\beta})$ and $\bar{B} \in (\mathcal{O}_X^{\oplus\beta}, \mathcal{O}_X(\alpha_1, \ldots, \alpha_t)^{\oplus\gamma})$ □

Corollary on existence

D Maingi [GOAL]

Theorem

Let $X = \mathbf{P}^{2n+1} \times \cdots \times \mathbf{P}^{2n+1}$ and $\mathscr{L} = \mathcal{O}_X(1, \cdots, 1)$ an ample line bundle. Denote by $N = h^0(\mathcal{O}_X(1, \cdots, 1)) - 1$. Then there exists a linear monad $M_\bullet$ on X of the form

$$M_\bullet : 0 \to \mathcal{O}_X(-1, \cdots, -1)^{\oplus\alpha} \xrightarrow[f]{} \mathcal{O}_X^{\oplus\beta} \xrightarrow[g]{} \mathcal{O}_X(1, \cdots, 1)^{\oplus\gamma} \to 0$$

if atleast one of the following is satified

- a. $\beta \geq 2\gamma + N - 1,$ *and* $\beta \geq \alpha + \gamma,$
- b. $\beta \geq \alpha + \gamma + N,$ *where* α, β, γ *be positive integers.*

Explicit monad construction via matrices

Let $\psi : X = \mathbf{P}^1 \times \cdots \times \mathbf{P}^1 \longrightarrow \mathbf{P}^{N=2n+1}$ be the Segre embedding which is defined as follows:

Explicit monad construction via matrices

Let $\psi : X = \mathbf{P}^1 \times \cdots \times \mathbf{P}^1 \longrightarrow \mathbf{P}^{N=2n+1}$ be the Segre embedding which is defined as follows:
$[\alpha_{10} : \alpha_{11}][\alpha_{20} : \alpha_{21}] : \ldots : [\alpha_{m0} : \alpha_{m1}] \hookrightarrow [x_0 : \cdots : x_n : y_0 : \ldots : y_n]$.

Explicit monad construction via matrices

Let $\psi : X = \mathbf{P}^1 \times \cdots \times \mathbf{P}^1 \longrightarrow \mathbf{P}^{N=2n+1}$ be the Segre embedding which is defined as follows:
$[\alpha_{10} : \alpha_{11}][\alpha_{20} : \alpha_{21}] : \ldots : [\alpha_{m0} : \alpha_{m1}] \hookrightarrow [x_0 : \cdots : x_n : y_0 : \ldots : y_n]$.
First note that since we are taking m copies of $\mathbf{P}^1$ then we have
$N = 2^m - 1 = 2^m - 2 + 1 = 2(2^{m-1} - 1) + 1 = 2n + 1$

Explicit monad construction via matrices

Let $\psi : X = \mathbf{P}^1 \times \cdots \times \mathbf{P}^1 \longrightarrow \mathbf{P}^{N=2n+1}$ be the Segre embedding which is defined as follows:
$[\alpha_{10} : \alpha_{11}][\alpha_{20} : \alpha_{21}] : \ldots : [\alpha_{m0} : \alpha_{m1}] \hookrightarrow [x_0 : \cdots : x_n : y_0 : \ldots : y_n]$.
First note that since we are taking m copies of $\mathbf{P}^1$ then we have
$N = 2^m - 1 = 2^m - 2 + 1 = 2(2^{m-1} - 1) + 1 = 2n + 1$
Thus from Fløystad, there exists a linear monad

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^{2n+1}}(-1)^{\oplus k} \xrightarrow[A]{} \mathcal{O}_{\mathbf{P}^{2n+1}}^{\oplus 2n+2k} \xrightarrow[B]{} \mathcal{O}_{\mathbf{P}^{2n+1}}(1)^{\oplus k} \longrightarrow 0$$

Explicit monad construction via matrices

Let $\psi : X = \mathbf{P}^1 \times \cdots \times \mathbf{P}^1 \longrightarrow \mathbf{P}^{N=2n+1}$ be the Segre embedding which is defined as follows:

$[\alpha_{10} : \alpha_{11}][\alpha_{20} : \alpha_{21}] : \ldots : [\alpha_{m0} : \alpha_{m1}] \hookrightarrow [x_0 : \cdots : x_n : y_0 : \ldots : y_n]$.

First note that since we are taking m copies of $\mathbf{P}^1$ then we have

$N = 2^m - 1 = 2^m - 2 + 1 = 2(2^{m-1} - 1) + 1 = 2n + 1$

Thus from Fløystad, there exists a linear monad

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^{2n+1}}(-1)^{\oplus k} \xrightarrow[A]{} \mathcal{O}_{\mathbf{P}^{2n+1}}^{\oplus 2n+2k} \xrightarrow[B]{} \mathcal{O}_{\mathbf{P}^{2n+1}}(1)^{\oplus k} \longrightarrow 0$$

From which we induce a monad on $X = \mathbf{P}^1 \times \cdots \times \mathbf{P}^1$

$$0 \to \mathcal{O}_X(-1, \cdots, -1)^{\oplus k} \xrightarrow[\overline{A}]{} \mathcal{O}_X^{\oplus 2n \oplus 2k} \xrightarrow[\overline{B}]{} \mathcal{O}_X(1, \cdots, 1)^{\oplus k} \to 0$$

Explicit monad construction via matrices

Let $\psi : X = \mathbf{P}^1 \times \cdots \times \mathbf{P}^1 \longrightarrow \mathbf{P}^{N=2n+1}$ be the Segre embedding which is defined as follows:
$[\alpha_{10} : \alpha_{11}][\alpha_{20} : \alpha_{21}] : \ldots : [\alpha_{m0} : \alpha_{m1}] \hookrightarrow [x_0 : \cdots : x_n : y_0 : \ldots : y_n]$.
First note that since we are taking m copies of $\mathbf{P}^1$ then we have
$N = 2^m - 1 = 2^m - 2 + 1 = 2(2^{m-1} - 1) + 1 = 2n + 1$
Thus from Fløystad, there exists a linear monad

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^{2n+1}}(-1)^{\oplus k} \xrightarrow[A]{} \mathcal{O}_{\mathbf{P}^{2n+1}}^{\oplus 2n+2k} \xrightarrow[B]{} \mathcal{O}_{\mathbf{P}^{2n+1}}(1)^{\oplus k} \longrightarrow 0$$

From which we induce a monad on $X = \mathbf{P}^1 \times \cdots \times \mathbf{P}^1$

$$0 \to \mathcal{O}_X(-1, \cdots, -1)^{\oplus k} \xrightarrow[\overline{A}]{} \mathcal{O}_X^{\oplus 2n \oplus 2k} \xrightarrow[\overline{B}]{} \mathcal{O}_X(1, \cdots, 1)^{\oplus k} \to 0$$

by giving the morphisms $\overline{A}$ and $\overline{B}$ with $\overline{B} \cdot \overline{A} = 0$ and $\overline{A}$ and $\overline{B}$ are of maximal rank.

Explicit monad construction via matrices

Let $\psi : X = \mathbf{P}^1 \times \cdots \times \mathbf{P}^1 \longrightarrow \mathbf{P}^{N=2n+1}$ be the Segre embedding which is defined as follows:
$[\alpha_{10} : \alpha_{11}][\alpha_{20} : \alpha_{21}] : \ldots : [\alpha_{m0} : \alpha_{m1}] \hookrightarrow [x_0 : \cdots : x_n : y_0 : \ldots : y_n]$.
First note that since we are taking m copies of $\mathbf{P}^1$ then we have
$N = 2^m - 1 = 2^m - 2 + 1 = 2(2^{m-1} - 1) + 1 = 2n + 1$
Thus from Fløystad, there exists a linear monad

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^{2n+1}}(-1)^{\oplus k} \xrightarrow[A]{} \mathcal{O}_{\mathbf{P}^{2n+1}}^{\oplus 2n+2k} \xrightarrow[B]{} \mathcal{O}_{\mathbf{P}^{2n+1}}(1)^{\oplus k} \longrightarrow 0$$

From which we induce a monad on $X = \mathbf{P}^1 \times \cdots \times \mathbf{P}^1$

$$0 \to \mathcal{O}_X(-1, \cdots, -1)^{\oplus k} \xrightarrow[\overline{A}]{} \mathcal{O}_X^{\oplus 2n \oplus 2k} \xrightarrow[\overline{B}]{} \mathcal{O}_X(1, \cdots, 1)^{\oplus k} \to 0$$

by giving the morphisms $\overline{A}$ and $\overline{B}$ with $\overline{B} \cdot \overline{A} = 0$ and $\overline{A}$ and $\overline{B}$ are of maximal rank.
From A and B whose entries are $x_0, \cdots, x_n, y_0, \cdots, y_n$ the homogeneous coordinates on $\mathbf{P}^{2n+1}$ we give the correspondence for the the Segre embedding using the following table:

Explicit monad construction via matrices

$homog.coord.\ \ on\ \ \mathbf{P}^{2n+1}$	$representation\ homog.coord.\ \ on\ \ X$
x_0	$a_{0000\cdots 0000}$
x_1	$a_{0000\cdots 0001}$
x_2	$a_{0000\cdots 0010}$
$\vdots$	$\vdots$
x_{n-1}	$a_{0111\cdots 1110}$
x_n	$a_{0111\cdots 1111}$
y_0	$a_{1000\cdots 0000}$
y_1	$a_{1000\cdots 0001}$
y_2	$a_{1000\cdots 0010}$
$\vdots$	$\vdots$
y_{n-1}	$a_{1111\cdots 1110}$
y_n	$a_{1111\cdots 1111}$

Explicit monad construction via matrices

$homog.coord.\ \ on\ \ \mathbf{P}^{2n+1}$	$homog.coord.\ \ on\ \ X$
$a_{0000\cdots0000}$	$\alpha_{10}\alpha_{20}\alpha_{30}\alpha_{40}\cdots\alpha_{(m-3)0}\alpha_{(m-2)0}\alpha_{(m-1)0}\alpha_{m0}$
$a_{0000\cdots0001}$	$\alpha_{10}\alpha_{20}\alpha_{30}\alpha_{40}\cdots\alpha_{(m-3)0}\alpha_{(m-2)0}\alpha_{(m-1)0}\alpha_{m1}$
$a_{0000\cdots0010}$	$\alpha_{10}\alpha_{20}\alpha_{30}\alpha_{40}\cdots\alpha_{(m-3)0}\alpha_{(m-2)0}\alpha_{(m-1)1}\alpha_{m0}$
$\vdots$	$\vdots$
$a_{0111\cdots1110}$	$\alpha_{10}\alpha_{21}\alpha_{31}\alpha_{41}\cdots\alpha_{(m-3)1}\alpha_{(m-2)1}\alpha_{(m-1)1}\alpha_{m0}$
$a_{0111\cdots1111}$	$\alpha_{10}\alpha_{21}\alpha_{31}\alpha_{41}\cdots\alpha_{(m-3)1}\alpha_{(m-2)1}\alpha_{(m-1)1}\alpha_{m1}$
$a_{1000\cdots0000}$	$\alpha_{11}\alpha_{20}\alpha_{30}\alpha_{40}\cdots\alpha_{(m-3)0}\alpha_{(m-2)0}\alpha_{(m-1)0}\alpha_{m0}$
$a_{1000\cdots0001}$	$\alpha_{11}\alpha_{20}\alpha_{30}\alpha_{40}\cdots\alpha_{(m-3)0}\alpha_{(m-2)0}\alpha_{(m-1)0}\alpha_{m1}$
$a_{1000\cdots0010}$	$\alpha_{11}\alpha_{20}\alpha_{30}\alpha_{40}\cdots\alpha_{(m-3)0}\alpha_{(m-2)0}\alpha_{(m-1)1}\alpha_{m0}$
$\vdots$	$\vdots$
$a_{1111\cdots1110}$	$\alpha_{11}\alpha_{21}\alpha_{31}\alpha_{41}\cdots\alpha_{(m-3)1}\alpha_{(m-2)1}\alpha_{(m-1)1}\alpha_{m0}$
$a_{1111\cdots1111}$	$\alpha_{11}\alpha_{21}\alpha_{31}\alpha_{41}\cdots\alpha_{(m-3)1}\alpha_{(m-2)1}\alpha_{(m-1)1}\alpha_{m1}$

Explicit monad construction via matrices

Specifically we define $\overline{A}$ and $\overline{B}$ as follows

$$\overline{B} := \left[\begin{array}{cccccc|cc} a_{0000\cdots0000} & \cdots & & a_{0111\cdots1111} & & & a_{1000\cdots0000} & \cdots \\ & \ddots & & \ddots & & & & \ddots \\ & & a_{0000\cdots0000} & & \cdots & a_{0111\cdots1111} & & \end{array}\right.$$

and

$$\overline{A} := \left[\begin{array}{cccc} -a_{1000\cdots0000} \cdots & -a_{1111\cdots1111} & & \\ & \ddots & \ddots & \\ & & -a_{1000\cdots0000} \cdots & -a_{1111\cdots1111} \\ \hline a_{0000\cdots0000} \cdots & a_{0111\cdots1111} & & \\ & \ddots & \ddots & \\ & & a_{0000\cdots0000} \cdots & a_{0111\cdots1111} \end{array}\right]$$

We note that

1. $\overline{B} \cdot \overline{A} = 0$ and
2. The matrices $\overline{B}$ and $\overline{A}$ have maximal rank

Vector bundles

Suppose the ambient space is $X = \mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$ then $\mathrm{Pic}(X) \simeq \mathbb{Z}^n$.

Vector bundles

Suppose the ambient space is $X = \mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$ then $\mathrm{Pic}(X) \simeq \mathbb{Z}^n$. We shall denote by g_i for $i = 1 \cdots, n$ the generators of the Picard group of X, $\mathrm{Pic}(X)$.

Vector bundles

Suppose the ambient space is $X = \mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$ then $\mathrm{Pic}(X) \simeq \mathbb{Z}^n$. We shall denote by g_i for $i = 1 \cdots, n$ the generators of the Picard group of X, $\mathrm{Pic}(X)$.
Denote by $\mathcal{O}_X(g_1, \cdots, g_n) := {p_1}^*\mathcal{O}_{\mathbf{P}^{a_1}}(g_1) \otimes \cdots \otimes {p_n}^*\mathcal{O}_{\mathbf{P}^{a_n}}(g_n)$, where p_i for $i = 1, \cdots, n$ are natural projections from X onto $\mathbf{P}^{a_i}$.

Vector bundles

Suppose the ambient space is $X = \mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$ then $\mathrm{Pic}(X) \simeq \mathbb{Z}^n$. We shall denote by g_i for $i = 1 \cdots, n$ the generators of the Picard group of X, $\mathrm{Pic}(X)$.
Denote by $\mathcal{O}_X(g_1, \cdots, g_n) := {p_1}^* \mathcal{O}_{\mathbf{P}^{a_1}}(g_1) \otimes \cdots \otimes {p_n}^* \mathcal{O}_{\mathbf{P}^{a_n}}(g_n)$, where p_i for $i = 1, \cdots, n$ are natural projections from X onto $\mathbf{P}^{a_i}$.
For any line bundle $\mathscr{L} = \mathcal{O}_X(g_1, g_2, \cdots, g_n)$ on X and a vector bundle E, we write $E(g_1, g_2, \cdots, g_n) = E \otimes \mathcal{O}_X(g_1, g_2, \cdots, g_n)$ and $(g_1, g_2, \cdots, g_n) := g_1[h_1 \times \mathbf{P}^{a_1}] + \cdots + g_n[\mathbf{P}^{a_n} \times h_n]$ representing its corresponding divisor.

Vector bundles

Suppose the ambient space is $X = \mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$ then $\mathrm{Pic}(X) \simeq \mathbb{Z}^n$. We shall denote by g_i for $i = 1 \cdots, n$ the generators of the Picard group of X, $\mathrm{Pic}(X)$.
Denote by $\mathcal{O}_X(g_1, \cdots, g_n) := {p_1}^*\mathcal{O}_{\mathbf{P}^{a_1}}(g_1) \otimes \cdots \otimes {p_n}^*\mathcal{O}_{\mathbf{P}^{a_n}}(g_n)$, where p_i for $i = 1, \cdots, n$ are natural projections from X onto $\mathbf{P}^{a_i}$.
For any line bundle $\mathscr{L} = \mathcal{O}_X(g_1, g_2, \cdots, g_n)$ on X and a vector bundle E, we write $E(g_1, g_2, \cdots, g_n) = E \otimes \mathcal{O}_X(g_1, g_2, \cdots, g_n)$ and $(g_1, g_2, \cdots, g_n) := g_1[h_1 \times \mathbf{P}^{a_1}] + \cdots + g_n[\mathbf{P}^{a_n} \times h_n]$ representing its corresponding divisor.
The normalization of E on X with respect to $\mathscr{L}$ is defined as follows:

Vector bundles

Suppose the ambient space is $X = \mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$ then $\mathrm{Pic}(X) \simeq \mathbb{Z}^n$. We shall denote by g_i for $i = 1 \cdots, n$ the generators of the Picard group of X, $\mathrm{Pic}(X)$.
Denote by $\mathcal{O}_X(g_1, \cdots, g_n) := {p_1}^*\mathcal{O}_{\mathbf{P}^{a_1}}(g_1) \otimes \cdots \otimes {p_n}^*\mathcal{O}_{\mathbf{P}^{a_n}}(g_n)$, where p_i for $i = 1, \cdots, n$ are natural projections from X onto $\mathbf{P}^{a_i}$.
For any line bundle $\mathscr{L} = \mathcal{O}_X(g_1, g_2, \cdots, g_n)$ on X and a vector bundle E, we write $E(g_1, g_2, \cdots, g_n) = E \otimes \mathcal{O}_X(g_1, g_2, \cdots, g_n)$ and $(g_1, g_2, \cdots, g_n) := g_1[h_1 \times \mathbf{P}^{a_1}] + \cdots + g_n[\mathbf{P}^{a_n} \times h_n]$ representing its corresponding divisor.
The normalization of E on X with respect to $\mathscr{L}$ is defined as follows: Set $d = \deg_{\mathscr{L}}(\mathcal{O}_X(1, 0, \cdots, 0))$, since $\deg_{\mathscr{L}}(E(-k_E, 0, \cdots, 0)) = \deg_{\mathscr{L}}(E) - nk \cdot (E)$ there is a unique integer $k_E := \lceil \mu_{\mathscr{L}}(E)/d \rceil$ such that $1 - d.(E) \leq \deg_{\mathscr{L}}(E(-k_E, 0, \cdots, 0)) \leq 0$.
The twisted bundle $E_{\mathscr{L}-norm} := E(-k_E, 0, \cdots, 0)$ is called the $\mathscr{L}$-normalization of E.

Vector bundles

Suppose the ambient space is $X = \mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$ then $\operatorname{Pic}(X) \simeq \mathbb{Z}^n$. We shall denote by g_i for $i = 1 \cdots, n$ the generators of the Picard group of X, $\operatorname{Pic}(X)$.

Denote by $\mathcal{O}_X(g_1, \cdots, g_n) := {p_1}^* \mathcal{O}_{\mathbf{P}^{a_1}}(g_1) \otimes \cdots \otimes {p_n}^* \mathcal{O}_{\mathbf{P}^{a_n}}(g_n)$, where p_i for $i = 1, \cdots, n$ are natural projections from X onto $\mathbf{P}^{a_i}$.

For any line bundle $\mathscr{L} = \mathcal{O}_X(g_1, g_2, \cdots, g_n)$ on X and a vector bundle E, we write $E(g_1, g_2, \cdots, g_n) = E \otimes \mathcal{O}_X(g_1, g_2, \cdots, g_n)$ and $(g_1, g_2, \cdots, g_n) := g_1[h_1 \times \mathbf{P}^{a_1}] + \cdots + g_n[\mathbf{P}^{a_n} \times h_n]$ representing its corresponding divisor.

The normalization of E on X with respect to $\mathscr{L}$ is defined as follows: Set $d = \deg_{\mathscr{L}}(\mathcal{O}_X(1, 0, \cdots, 0))$, since $\deg_{\mathscr{L}}(E(-k_E, 0, \cdots, 0)) = \deg_{\mathscr{L}}(E) - nk \cdot (E)$ there is a unique integer $k_E := \lceil \mu_{\mathscr{L}}(E)/d \rceil$ such that $1 - d.(E) \leq \deg_{\mathscr{L}}(E(-k_E, 0, \cdots, 0)) \leq 0$.

The twisted bundle $E_{\mathscr{L}-norm} := E(-k_E, 0, \cdots, 0)$ is called the $\mathscr{L}$-normalization of E.

Lastly, the linear functional $\delta_{\mathscr{L}}$ on $\mathbb{Z}^n$ is defined as $\delta_{\mathscr{L}}(p_1, p_2, \cdots, p_n) := \deg_{\mathscr{L}} \mathcal{O}_X(p_1, p_2, \cdots, p_n)$.

Hoppe criterion on stability

Lemma

Let X be a polycyclic variety with Picard number n, let $\mathscr{L}$ be an ample line bundle and let E be a rank $r > 1$ holomorphic vector bundle over X. If $H^0(X, (\bigwedge^q E)_{\mathscr{L}-norm}(p_1, \cdots, p_n)) = 0$ for $1 \leq q \leq r-1$ and every $(p_1, \cdots, p_n) \in \mathbb{Z}^n$ such that $\delta_{\mathscr{L}} \leq 0$ then E is $\mathscr{L}$-stable.

Theorem

Let F be a vector bundle on $X = \mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$ defined by the short exact sequence

$$0 \longrightarrow F \longrightarrow \mathcal{O}_X^{\oplus \beta} \xrightarrow[g]{} \mathcal{O}_X(\alpha_1, \cdots, \alpha_t)^{\oplus \gamma} \longrightarrow 0$$

then F is stable for an ample line bundle $\mathscr{L} = \mathcal{O}_X(\alpha_1, \cdots, \alpha_t)$

Proof.

$$H^0(X, \bigwedge^q F(-p_1, \cdots, -p_n)) = 0 \text{ for all } \sum_{i=1}^{n} p_i > 0 \text{ and } 1 \leq q \leq (F) - 1$$

□

Theorem

Let $X = \mathbf{P}^{a_1} \times \cdots \times \mathbf{P}^{a_n}$, *then the cohomology bundle* E *associated to the monad*

$$0 \to \mathcal{O}_X(-\alpha_1, \cdots, -\alpha_t)^{\oplus\alpha} \xrightarrow[f]{} \mathcal{O}_X^{\oplus\beta} \xrightarrow[g]{} \mathcal{O}_X(\alpha_1, \cdots, \alpha_t)^{\oplus\gamma} \to 0$$

of rank $\beta - \alpha - \gamma$ *is simple.*

References

1. V Ancona, G Ottaviani: *Stability of special instanton Bundles on* $\mathbf{P}^{2n+1}$ Transactions of the American Mathematical Society 341 (1994) 677 - 693.
2. G Fløystad: *Monads on a Projective Space.* Comm Algebra, 28 (2000), 5503 - 5516.
3. R Hartshone: *Algebraic Geometry*, Springer, 1977
4. G Horrocks, D Mumford: *A rank 2 vector bundle on* $\mathbf{P}^4$ *with 15000 symmetries, Topology* Topology 12 (1973) 63-81.
5. Maingi D. Vector Bundles of low rank on a multiprojective space. Le Matematiche. Vol. LXIX (2014) - Fasc. II. pp 31-41. doi: 10.4418/2014.69.2.4.
6. Maingi D (2021). Indecomposable Vector Bundles associated to Monads on Cartesian products of projective spaces. Turkish Journal of Mathematics. Vol. 45: No. 5. Article 17. Pages 2126-2139. doi: 10.3906/mat-2101-6
7. Maingi D. Monads on multiprojective Products of Projective Spaces. Manuscripta Mathematica (2022).doi: