N-foci balls in hyperbolic geometry

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Properties of the hyperbolic plane

Let's suppose that $\mathbb{H}^2 = \{(x, y) \mid y > 0\}$ is an upper half–plane with the Riemannian metric $\dfrac{dx^2 + dy^2}{y^2}$. It is called a hyperbolic plane and has a constant negative Gaussian curvature -1. Besides, $\mathbb{H}^2$ is a Hadamard space, so it is a complete Riemannian manifold of nonpositive sectional curvature.

Properties of the hyperbolic plane

Between two any points $x, y \in \mathbb{H}^2$ there is a unique geodesic $\sigma_{x,y}$. So we can define a notion of a geodesically convex (or just convex) set in hyperbolic plane — it is a set that for two arbitrary points x and y of its $\sigma_{x,y}$ belongs to this set. Particularly, the mapping

$$\rho : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{R}, \rho(x, y) = \ell(\sigma_{x,y}), x, y \in \mathbb{H}^2,$$

where ℓ denotes a length of curve in $\mathbb{H}^2$, satisfies all the axioms of metric space.

N-foci balls in the hyperbolic plane

Let's fix in $\mathbb{H}^2$ any mutually distinct points $x_1, \ldots, x_N$, where $N \in \mathbb{N}$, and such positive numbers $w_1, \ldots, w_N, a$ that $\sum\limits_{k=1}^{N} w_k = 1$.

Definition

Open weighted N-foci ball, or weighted N-foci ball, is a set

$$A = \{x \in \mathbb{H}^2 \mid w_1\, \rho(x, x_1) + \cdots + w_N\, \rho(x, x_N) < a\}, \qquad (1)$$

where $x_1, \ldots, x_N$ are called foci of the weighted N-foci ball, a is called a radius of the weighted N-foci ball, $w_1, \ldots, w_N$ are called weights of the foci $x_1, \ldots, x_N$.

N-foci balls in the hyperbolic plane

We can define closed weighted N-foci balls the same way, having replaced the symbol "$<$" by the symbol "$\leq$" in the formula (1). We can also define weighted N-ellipses the same way, having replaced the symbol "$<$" by the symbol "$=$" in the formula (1).

Convexity of hyperbolic balls

Theorem

Circles in the hyperbolic plane geometrically coincide with Euclidean circles.

Actually, if we take one circle with a hyperbolic centre in $z_0 \in \mathbb{C}$ and map it with the function

$$f(z) = \frac{z - z_0}{z - \overline{z_0}},\ z \in \mathbb{C},$$

we will obtain an image of this circle in the model of hyperbolic disk where $f(z_0) = 0$. Besides, all the rotations around the coordinate centre are isometries of the hyperbolic disk. So, we have a standard Euclidean circle, and the initial hyperbolic circle is a Euclidean one too according to the properties of Möbius transformations.

Convexity of hyperbolic balls

Theorem

Any open ball of a positive radius in the hyperbolic plane is geodesically convex.

The first approach. One can map the circle from hyperbolic disk with its center in the coordinate center to the Poincaré–Klein model of the hyperbolic plane where all the geodesics are straight like in a Euclidean circle. Then we get a Euclidean circle with its center in the coordinate center as image of the researched hyperbolic circle again and it is easy to see its convexity.

Convexity of hyperbolic balls

The second approach. We know that we can research a usual Euclidean circle in the upper half–plane model. All the geodesics there are either vertical straight lines or arcs of circles with their centers on x-axis.
There is a simple inversion which maps considered model isometrically in itself so that the circle is mapped into another circle where one concrete geodesic as an arc of a circle becomes a straight vertical line. It allows to see convexity of the circle too.

Definition

We will call a parametrization $\gamma : [0,1] \to \mathbb{H}^2$ of the geodesics between points a and b in $\mathbb{H}^2$, $\gamma(0) = a$, $\gamma(1) = b$, standard, if for all $\alpha \in (0;1)$ the equality

$$\rho(a, \gamma(\alpha)) = \alpha\ell$$

holds. Here ℓ denotes a length of the appropriate geodesics.

Convex functions in Riemannian manifolds

Definition

A function $f : \mathbb{H}^2 \to \mathbb{R}$ is called convex in a convex set $A \subset \mathbb{H}^2$, if for arbitrary points $x_1, x_2 \in A$ and a standard parametrization $\gamma : [0,1] \to \mathbb{H}^2$ of the geodesics between them, $\gamma(0) = x_2$, $\gamma(1) = x_1$, next inequality holds:

$$\forall \alpha \in [0,1] : f(\gamma(\alpha)) \leq \alpha f(x_1) + (1-\alpha) f(x_2). \qquad (2)$$

Convex functions in Riemannian manifolds

Theorem

If A is a convex subset of the Riemannian manifold M, and $f_k : A \to \mathbb{R}$, $k = \overline{1, n}$, $n \in \mathbb{N}$, are convex in A, then the function

$$f(x) = \sum_{k=1}^{n} w_k f_k(x), x \in A,$$

where $w_1, \ldots, w_n$ are positive, is convex in A too.

Convex functions in Riemannian manifolds

Theorem

If a function $f : M \to \mathbb{R}$, where M is a Riemannian manifold, is convex in a convex set $A \subset M$, then for all $a \in \mathbb{R}$ the sets

$$A_1 := A \cap \{x \in M | f(x) \leq a\}, \quad A_2 := A \cap \{x \in M | f(x) < a\}.$$

are convex.

Distance function in the hyperbolic plane

Let's fix any point $x_0 \in \mathbb{H}^2$ and define the distance function for it:

$$f : \mathbb{H}^2 \to \mathbb{R},\ f(x) = \rho(x, x_0),\ x \in \mathbb{H}^2.$$

Theorem

The distance function f is convex in the hyperbolic plane $\mathbb{H}^2$.

Distance function in the hyperbolic plane

It is known, that such a function is convex in any Hadamard space. In this work we got a direct proof of convexity of f for the case of the hyperbolic plane. Moreover, here we made sure that topological and analytical methods often can give more satisfying answer to similar problems than purely geometrical ones.

Distance function in the hyperbolic plane

We didn't research hyperbolic spaces of older dimensions in this work, but it is worth highlighting that the same result about convexity of the distance function is also right for their case, because they are Hadamard spaces as well. But we will consider now only the case of the hyperbolic plane to look through the general idea of the proof.

Distance function in the hyperbolic plane

Having used some isometries of the hyperbolic plane, especially inversion, it was stated that it is sufficient to discover the function f at the vertical geodesic in the imaginary axis, so that its lower limit equals to i.

Then, let's assume that $\gamma : [0,1] \to \mathbb{H}^2$ is a standard parametrization of the geodesic. If x_0 is in the imaginary axis, it is enough to prove the convexity of the function

$$h(\alpha) = f(\gamma(\alpha)) = \left|\ln \frac{a_2^\alpha}{q}\right|, \alpha \in \mathbb{R},$$

in $[0,1]$.

Distance function in the hyperbolic plane

Otherwise, it is enough to prove the convexity of the function

$$h(\alpha) = f(\gamma(\alpha)) = \ln\left(\frac{2}{1 - \frac{|a_2^\alpha i - p - iq|}{|a_2^\alpha i + p + iq|}} - 1\right), \ \alpha \in \mathbb{R}$$

in $[0, 1]$, where $x_0 = p + iq$.

Convexity of N-foci balls

From the obtained result implies the next theorem.

Theorem

All open and closed weighted N-foci balls are geodesically convex sets in the hyperbolic plane $\mathbb{H}^2$.