On the asymptotic behavior of solutions to nonlinear Beltrami equation

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Beltrami equation

Let $\mathbb{C}$ be the complex plane. In the complex notation $f = u + iv$ and $z = x + iy$, the *Beltrami equation* in a domain $G \subset \mathbb{C}$ has the form

$$f_{\overline{z}} = \mu(z) f_z, \tag{1}$$

where $\mu : G \to \mathbb{C}$ is a measurable function and

$$f_{\overline{z}} = \frac{1}{2}(f_x + i f_y), \qquad f_z = \frac{1}{2}(f_x - i f_y)$$

are formal derivatives of f in $\overline{z}$ and z, while f_x and f_y are partial derivatives of f in the variables x and y, respectively.

Nonlinear Beltrami equation

Let $\sigma : G \to \mathbb{C}$ be a measurable function and $m \geqslant 0$. We consider the following equation written in the polar coordinates (r, θ) :

$$f_r = \sigma(re^{i\theta})\,|f_\theta|^m\, f_\theta, \tag{2}$$

where f_θ and f_r are the partial derivatives of f by θ and r, respectively. The equations of this type were studied in the works [1]–[6].

Nonlinear Beltrami equation

Applying the relations between these derivatives and the formal derivatives

$$\text{(3)} \qquad r f_r = z f_z + \bar{z} f_{\bar{z}}\,, \qquad f_\theta = i(z f_z - \bar{z} f_{\bar{z}})\,,$$

one can rewrite the equation (2) in the Cartesian form:

$$\text{(4)} \qquad f_{\bar{z}} = \frac{z}{\bar{z}} \frac{\widetilde{\sigma}(z)\,|z f_z - \bar{z} f_{\bar{z}}|^m - 1}{\widetilde{\sigma}(z)\,|z f_z - \bar{z} f_{\bar{z}}|^m + 1}\, f_z\,,$$

where $\widetilde{\sigma}(z) = i\sigma(z)|z|$.

Nonlinear Beltrami equation

Under $\mathrm{m} = 0$, the equation (4) reduces to the standard linear Beltrami equation (1) with the complex coefficient

$$\mu(\mathrm{z}) = \frac{\mathrm{z}}{\bar{\mathrm{z}}} \frac{\mathrm{i}\sigma(\mathrm{z})\,|\mathrm{z}| - 1}{\mathrm{i}\sigma(\mathrm{z})\,|\mathrm{z}| + 1}\,.$$

Picking $\mathrm{m} = 0$ and $\sigma = -\mathrm{i}/|\mathrm{z}|$ in (4), we arrive at the classical Cauchy-Riemann system. For $\mathrm{m} > 0$ the equation (4) provides a partial case of the general nonlinear system of equations (7.33) given in [7].

Nonlinear Beltrami equation

Next, we consider an equation of another type, namely

$$f_\theta = \sigma(re^{i\theta})\,|f_r|^m\, f_r. \tag{5}$$

Applying the relations (3), one can rewrite the equation (5) by

$$f_{\bar z} = \frac{z}{\bar z}\,\frac{1 + i\sigma(z)\,|z|^{-m-1}|zf_z + \bar z f_{\bar z}|^m}{1 - i\sigma(z)\,|z|^{-m-1}|zf_z + \bar z f_{\bar z}|^m}\, f_z. \tag{6}$$

Assuming $m = 0$, the equation (6) also becomes the standard linear Beltrami equation (1) with

$$\mu(z) = \frac{z}{\bar z}\,\frac{1 + i\sigma(z)/|z|}{1 - i\sigma(z)/|z|}\,.$$

Choosing $m = 0$ and $\sigma = i|z|$ in (6), we arrive again at the classical Cauchy-Riemann system. Later on we assume that $m > 0$.

Regular homeomorphic solutions

A mapping $f\colon G\to\mathbb{C}$ is called *regular at a point* $z_0\in G$, if f has the total differential at this point and its Jacobian $J_f=|f_z|^2-|f_{\bar z}|^2$ does not vanish. A homeomorphism f of Sobolev class $W^{1,1}_{loc}$ is called *regular*, if $J_f>0$ a.e. By a *regular homeomorphic solution of the equation* (6) we call a regular homeomorphism $f\colon G\to\mathbb{C}$, which satisfies (6) a.e. in G.

Later on we use the following notations

$$B_r=\{z\in\mathbb{C}:|z|<r\}\,,\quad \mathbb{B}=\{z\in\mathbb{C}:|z|<1\}$$

and

$$\gamma_r=\{z\in\mathbb{C}:|z|=r\}\,,\quad \mathbb{A}(0,r_1,r_2)=\{z\in\mathbb{C}:r_1<|z|<r_2\}.$$

The area of set $f(B_r)$ we denote by $S_f(r)=|f(B_r)|$.

p-angular dilatation

Let $f : \mathbb{B} \to \mathbb{C}$ be a regular homeomorphism of the Sobolev class $W^{1,1}_{loc}$, and $p > 1$. By the p-*angular dilatation* of the mapping f with respect to the point $z_0 = 0$ we call a quantity

$$D_{p,f}(z) = D_{p,f}(re^{i\theta}) = \frac{|f_\theta(re^{i\theta})|^p}{r^p J_f(re^{i\theta})}, \tag{7}$$

where $z = re^{i\theta}$ and J_f is the Jacobian of f.
For $D_{p,f}(z)$ and $p > 1$, denote

$$d_{p,f}(r) = \left(\frac{1}{2\pi r} \int\limits_{\gamma_r} D_{p,f}^{\frac{1}{p-1}}(z)\, |dz| \right)^{p-1}. \tag{8}$$

Differential inequality

The following lemma provides a differential inequality for the area functional $S_f(r) = |f(B_r)|$.

Lemma

Let $f : \mathbb{B} \to \mathbb{C}$ *be a regular homeomorphism of the Sobolev class* $W^{1,1}_{loc}$ *that possesses the* N*-property, and* $p > 1$, $K > 0$. *If*

(9) $$d_{p,f}(r) \leqslant K \quad \textit{for a.a.} \quad r \in (0,1)\,,$$

then

(10) $$S'_f(r) \geqslant 2\pi^{\frac{2-p}{2}} K^{-1} r^{1-p} S_f^{\frac{p}{2}}(r)$$

for a.a. $r \in [0,1)$.

The area of the disk image

Lemma

Let $f : \mathbb{B} \to \mathbb{C}$ *be a regular homeomorphism of the Sobolev class* $W^{1,1}_{loc}$ *that possesses the* N*-property,* $1 < p < 2$ *and* $K > 0$*. If* $d_{p,f}(r) \leqslant K$ *for a.a.* $r \in (0,1)$*, then for* $r \in [0,1)$

$$|f(B_r)| \geqslant C(p,K)\, r^2\,, \tag{11}$$

where $C(p,K) = \pi K^{\frac{2}{p-2}}$*.*

Asymptotic behavior of regular homeomorphisms

Lemma

Let $f : \mathbb{B} \to \mathbb{C}$ *be a regular homeomorphism of the Sobolev class* $W^{1,1}_{loc}$ *that possesses the* N*-property and normalized by* $f(0) = 0$*, and* $1 < p < 2$, $K > 0$. *If* $d_{p,f}(r) \leqslant K$ *for a.a.* $r \in (0,1)$*, then*

$$\limsup_{z \to 0} \frac{|f(z)|}{|z|} \geqslant K^{-\frac{1}{2-p}}\,.$$

Asymptotic behavior of regular homeomorphisms

Theorem

Let $f:\mathbb{B}\to\mathbb{C}$ *be a regular homeomorphism of the Sobolev class* $W^{1,1}_{loc}$ *that possesses the* N*-property and normalized by* $f(0)=0$*, and* $1<p<2$*. Suppose that*

$$\kappa_0=\liminf_{\varepsilon\to 0}\left(\frac{1}{\pi\varepsilon^2}\int\limits_{B_\varepsilon} D_{p,f}^{\frac{1}{p-1}}(z)\,dx\,dy\right)^{p-1}.$$

1) If $\kappa_0\in(0,\infty)$*, then*

$$\limsup_{z\to 0}\frac{|f(z)|}{|z|}\geqslant c_p\,\kappa_0^{-\frac{1}{2-p}},$$

where c_p *is a positive constant depending on the parameter* p*.*

2) If $\kappa_0=0$*, then*

$$\limsup_{z\to 0}\frac{|f(z)|}{|z|}=\infty.$$

Asymptotic behavior of regular homeomorphic solutions

Theorem

Let $f:\mathbb{B}\to\mathbb{C}$ *be a regular homeomorphic solution of the equation (6) which belongs to Sobolev class* $W^{1,2}_{\mathrm{loc}}$*, and normalized by* $f(0)=0$*. Assume that* $C>0$ *and the coefficient* $\boldsymbol{\sigma}:\mathbb{B}\to\mathbb{C}$ *satisfies the following condition*

$$\int\limits_{\gamma_r}\frac{|\boldsymbol{\sigma}(z)|^{m+2}}{(\operatorname{Im}\boldsymbol{\sigma}(z))^{m+1}}\,|dz|\leqslant C\,r^2 \tag{12}$$

for a.a. $r\in(0,1)$*. Then*

$$\limsup_{z\to 0}\frac{|f(z)|}{|z|}\geqslant\left(\frac{2\pi}{C}\right)^{\frac{1}{m}}. \tag{13}$$

Asymptotic behavior of regular homeomorphic solutions

Corollary

Let $f : \mathbb{B} \to \mathbb{C}$ *be a regular homeomorphic solution of the equation (6) which belongs to Sobolev class* $W^{1,2}_{loc}$*, and normalized by* $f(0) = 0$ *and* $K > 0$*. Assume that the coefficient* $\sigma : \mathbb{B} \to \mathbb{C}$ *satisfies the following condition*

$$\frac{|\sigma(z)|^{m+2}}{(\operatorname{Im}\sigma(z))^{m+1}} \leqslant K\,|z| \tag{14}$$

for a.a. $z \in \mathbb{B}$*. Then*

$$\limsup_{z\to 0} \frac{|f(z)|}{|z|} \geqslant K^{-\frac{1}{m}}\,. \tag{15}$$

Example

Fix $k > 0$ and consider the equation

$$f_\theta = \frac{i}{k^m} r |f_r|^m f_r \tag{16}$$

in the unit disk $\mathbb{B}$. Let $f = kre^{i\theta}$. Obviously, the mapping f belongs to the Sobolev class $W^{1,2}(\mathbb{B})$. The partial derivatives of f with respect to θ and r are $f_\theta = kire^{i\theta}, f_r = k e^{i\theta}$ and $J_f(re^{i\theta}) = \frac{1}{r} \mathrm{Im}\left(\overline{f_r} f_\theta\right) = k^2 > 0$.
Now we show that the mapping $f = kre^{i\theta}$ is a solution of equation (16). Clearly, $\sigma = \frac{f_\theta}{|f_r|^m f_r} = \frac{i}{k^m} r$. Thus, (12) holds, since $\int\limits_{\gamma_r} \frac{|\sigma(z)|^{m+2}}{(\mathrm{Im}\,\sigma(z))^{m+1}} |dz| = C r^2$ where $C = \frac{2\pi}{k^m}$.
On the other hand, $\lim\limits_{z\to 0} \frac{|f(z)|}{|z|} = k$.

Asymptotic behavior of regular homeomorphic solutions

Theorem

Let $f:\mathbb{B}\to\mathbb{C}$ *be a regular homeomorphic solution of the equation* (6) *which belongs to Sobolev class* $W^{1,2}_{\rm loc}$, *and normalized by* $f(0)=0$. *Suppose that*

$$\sigma_0=\liminf_{\varepsilon\to 0}\frac{1}{\pi\varepsilon^2}\int\limits_{B_\varepsilon}\frac{|\sigma(z)|^{m+2}}{|z|\,(\mathrm{Im}\,\sigma(z))^{m+1}}\,dxdy.$$

1) If $\sigma_0\in(0,\infty)$, *then*

$$\limsup_{z\to 0}\frac{|f(z)|}{|z|}\geqslant c_m\,\sigma_0^{-\frac{1}{m}},$$

where c_m *is a positive constant depending on the parameter* m.
2) If $\sigma_0=0$, *then*

$$\limsup_{z\to 0}\frac{|f(z)|}{|z|}=\infty.$$

Asymptotic behavior of regular homeomorphic solutions

Example

Let $k > 0$ and $\alpha \in (1, m+1)$. Consider the equation

$$f_\theta = ikr^\alpha |f_r|^m f_r \tag{17}$$

in the unit disk $\mathbb{B}$. The mapping $f = k^{-\frac{1}{m}} \beta^{\frac{m+1}{m}} r^{\frac{m+1-\alpha}{m}} e^{i\theta}$, $\beta = \frac{m}{m+1-\alpha}$, belongs to the Sobolev class $W^{1,2}_{loc}(\mathbb{B})$. Its partial derivatives with respect to r and θ are $f_\theta = ik^{-\frac{1}{m}} \beta^{\frac{m+1}{m}} r^{\frac{m+1-\alpha}{m}} e^{i\theta}$, $f_r = k^{-\frac{1}{m}} \beta^{\frac{1}{m}} r^{\frac{1-\alpha}{m}} e^{i\theta}$.

Asymptotic behavior of regular homeomorphic solutions

Example

It is easy to see that the mapping $f = k^{-\frac{1}{m}}\beta^{\frac{m+1}{m}} r^{\frac{m+1-\alpha}{m}} e^{i\theta}$ is a regular homeomorphic solution of the equation (17). Clearly, $\sigma = \frac{f_\theta}{|f_r|^m f_r} = ikr^\alpha$. The condition $\sigma_0 = 0$ in previous theorem is fulfilled, since

$$\lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon^2}\int\limits_{B_\varepsilon}\frac{|\sigma(z)|^{m+2}}{|z|\,(\mathrm{Im}\,\sigma(z))^{m+1}}\,dxdy = 0.$$

By a direct calculation, $|f(z)|/|z| \to \infty$ as $z \to 0$.

References

1. A. Golberg, R. Salimov and M. Stefanchuk, *Asymptotic dilation of regular homeomorphisms*. Complex Anal. Oper. Theory, **13** (2019), no. 6, 2813–2827.
2. R. R. Salimov and M. V. Stefanchuk, *On the local properties of solutions of the nonlinear Beltrami equation*. J. Math. Sci., **248** (2020), 203–216.
3. A. Golberg and R. Salimov, *Nonlinear Beltrami equation*. Complex Var. Elliptic Equ., **65** (2020), no. 1, 6–21.
4. R. R. Salimov and M. V. Stefanchuk, *Logarithmic asymptotics of the nonlinear Cauchy-Riemann-Beltrami equation*. Ukr. Math. J., **73** (2021), 463–478.
5. R. R. Salimov, M. V. Stefanchuk. *Nonlinear Beltrami equation and asymptotics of its solution.* J. Math. Sci., **264** (2022), no. 4, 441–454.
6. R. Salimov, M. Stefanchuk. *Finite Lipschitzness of regular solutions to nonlinear Beltrami equation.* Complex Variables and Elliptic Equations, **69** (2024), no. 6, 913–923.

References

7. K. Astala, T. Iwaniec and G. Martin, *Elliptic partial differential equations and quasiconformal mappings in the plane*. Princeton Mathematical Series, 48. Princeton University Press, Princeton, NJ, 2009.

8. I. Petkov, R. Salimov, M. Stefanchuk. *Nonlinear Beltrami equation: lower estimates of Schwarz Lemma's type.* Canadian Mathematical Bulletin (2023). https://doi.org/10.4153/S0008439523000942