On the Brody hyperbolicity

Abdessami Jalled
(150 Pl. du Torrent, 38400 Saint-Martin-d'Hères, Office 178)
E-mail: `Abdessami.jalled@grenoble-inp.fr`

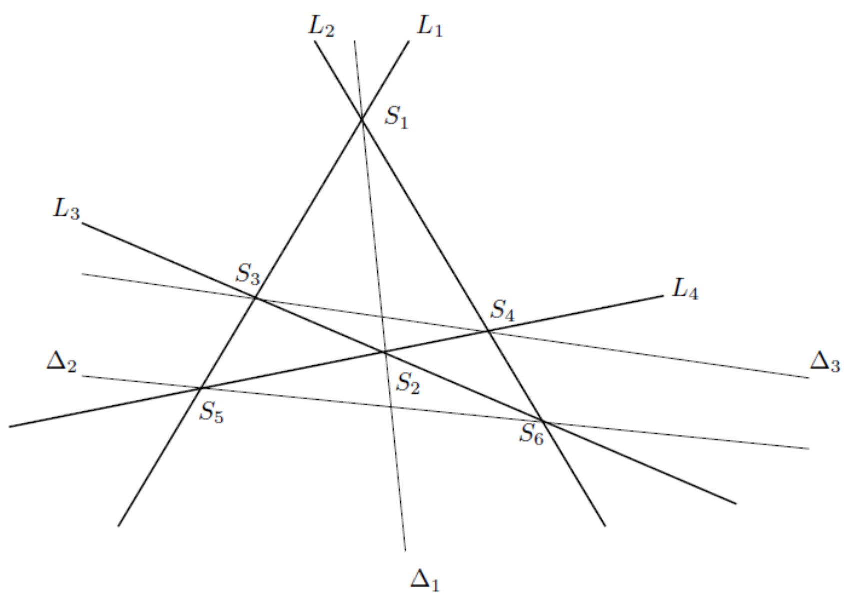
Definition 1.
Let $H_1, ..., H_m$, $m \geqslant 2n$, be a configuration of 2n hyperplanes in general position of $\mathbb{C}P^n$. We call diagonal, the line passing through the two points $\cap_{i\in I} H_i$ and $\cap_{j\in J} H_j$, where $|I| = |J| = n$ and $I \cap J = \emptyset$. Here $|I|$ denotes the cardinal of I.

Theorem 2.
Let $H_1, ..., H_{2n}$ be $(2n)$ projective hyperplanes in general position in $\mathbb{C}P^n$. Then there are $\frac{1}{2}C_{2n}^n$ diagonals $\Delta_1, ..., \Delta_{\frac{1}{2}C_{2n}^n}$ such that for any non constant holomorphic curve $f : \mathbb{C} \longrightarrow \mathbb{C}P^n \setminus \cup_{i=1}^{2n} H_i$, there exists $k_f \in \{1, ..., \frac{1}{2}C_{2n}^n\}$ such that $f(\mathbb{C}) \subset \Delta_{k_f}$.

Corollary 3. *(This is how we prove the Green Theorem).*
Any holomorphic curve that lies in the complement of $2n + 1$ hyperplanes in general position in $\mathbb{C}P^n$, is constant.

Theorem 4. *(E. Borel)*
Let $H = \cup_{i=1}^4 H_i$ a collection of complex projective lines in general position in $\mathbb{C}P^2$. Then any non constant map $f : \mathbb{C} \to \mathbb{C}P^2 \setminus H$, lies in one of the diagonales $(\Delta_i)_{i=1,2,3}$. Where Δ_i are the projective lines passing each through a double points of H.

Configuration $\bigcup_{1\leq k\leq 4} L_k$ and its diagonals



Theorem 5.
Let L_1, L_2, L_3, L_4 and L_5 complex hyperplanes in general position in $\mathbb{C}^3$, then for every holomorphic curve $G : \mathbb{C} \to \mathbb{C}^3$ such that $G(\mathbb{C}) \cap (\cup_{i=1}^5 L_i) = \emptyset$, there exists a complex line L in $\mathbb{C}^3$ such that $G(\mathbb{C}) \subset L$. Moreover, the complementary of five complex lines in $\mathbb{C}^3$ is not Brody hyperbolic. (This result is also true in higher dimension)
Remark: The projection of G into the complex projective space $\mathbb{C}P^2$ is constant.

Definition 6. For $n \geqslant 3$ and $\mathcal{L} = (L_1, ..., L_n)$ a family of real subspaces of $\mathbb{R}^6$ of real codimension 2. Then we say that $\mathcal{L}$ is in general position if for every 3-tuple (i, j, l) of distinct integers $i, j, l \in \{1, ..., n\}$,

$$Span_{\mathbb{R}}(L_i^{\perp}, L_j^{\perp}, L_l^{\perp}) = \mathbb{R}^6.$$

We note that if L is a real subspace in $\mathbb{R}^6$, then $L^{\perp}$ denotes the orthogonal complement of L.

Theorem 7. *Let L_1, L_2, L_3, L_4 be four complex lines in $\mathbb{C}^3$. Then there exists a real subspace L of $\mathbb{R}^6$, of real dimension four, such that (L, L_i, L_j) are in general position for all $j \neq i, \quad j, i \in \{1, ..., 4\}$, and there exists a non constant holomorphic curve $g : \mathbb{C} \to \mathbb{C}^3$, such that*

$$g(\mathbb{C}) \bigcap (\bigcup_{i=1}^{4} L_i \bigcup L) = \emptyset$$

ie, the complementary of this configuration in $\mathbb{C}^3$ is not Brody hyperbolic.
Remark: The projection of G into the complex projective space $\mathbb{C}P^2$ is not constant.

Here π denotes the canonical projection from $\mathbb{C}^3 \setminus \{0\}$ into $\mathbb{C}P^2$ and $\pi(g) := \pi \circ g$. Notice that $\pi(g)$ is well-defined since $g(\mathbb{C}) \subset \mathbb{C}^3 \setminus \{0\}$.

Theorem 8.
The complementary of five real subspaces $\tilde{L}_i$, $i = 1 \dots 5$ of real dimension 5 in $\mathbb{C}^3$ is Brody hyperbolic. That is to say that any holomorphic map $g : \mathbb{C} \to \mathbb{C}^3 \setminus \cup_{i=1}^5 \tilde{L}_i$ is constant.

References

[1] Haggui Jalled. ON THE BOREL THEOREM. *Matematiqki Bilten*, 45 (LXXI) No.1, 17–21. 2021
[2] B. Saleur. *Un théorème de Bloch presque complexe*, volume 46 of *Ann. Inst. Fourier (Grenoble)*. 64(2):401–428, 2014.
[3] A. Jalled. *ON THE BRODY HYPERBOLICITY OF THE COMPLEX SPACE C3 AVOIDING COMPLEX AND REAL HYPERPLANES*, *Ukrains'kyi Matematychnyi Zhurnal* . 2025.
[4] H. Fathi, A. Jalled. *Some remarks on a theorem of Green*, *Proceedings of the International Geometry Center*. Vol. 15, no. 3-4. pp. 177–183 (2022)