HOMOGENEOUS HOMODERIVATIONS ON GRADED ASSOCIATIVE RINGS

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As a fact, the concept of homogeneous derivations was introduced by Kanunnikov (2018)[1]. Let Λ be a G-graded ring. An additive mapping $\kappa\colon \Lambda \to \Lambda$ is called homogeneous derivation if:
(i) $\kappa(xy) = \kappa(x)y + x\kappa(y)$ for all $x, y \in \Lambda$.
(ii) $\kappa(r) \in H(\Lambda)$ for all $r \in H(\Lambda)$.
While the year 2000 came, a classical definition concerning of homoderivation was delivered via article [2], where an additive mapping is a homoderivation concerning Λ like from Λ to Λ, where Λ is a ring. In other expressions, (from a ring to itself) satisfy $\kappa(xy) = \kappa(x)\kappa(y) + \kappa(x)y + x\kappa(y)$ where x and y in Λ. Every homogeneous derivation is a derivation. However, the converse statement is not true, as there exist derivations that are not homogeneous.
Over the last 70 years, researchers have been interested in understanding the structure and commutativity of ring R using specific types of mappings called derivations. Various authors have widely studied this topic. Like [3]. In 1957, the study of commutativity of prime rings with derivations was initiated. Since then, the relationship between the commutativity of rings and the existence of specific types of derivations has attracted many researchers. The main result in this context is that a prime ring R with a nonzero centralizing derivation d must be a commutative ring. Graded rings have various applications in geometry and physics, and appear in various contexts, from elementary to advanced levels. Based on the rich heritage of ring theory, many researchers have attempted to extend and generalize various classical results to graded settings.
In this paper, Λ represents an associative ring with the center $Z(\Lambda)$, and G is an abelian group with identity element e. For $x, y \in \Lambda$, we write $[x, y]$ for Lie product $xy - yx$ and for a nonempty subset S for Λ, we write $C_\Lambda(S) = \{x \in \Lambda \mid [x, S] = 0\}$ for the centralizer of S in Λ. A ring Λ is G-graded if there is a family $\{\Lambda_g, g \in G\}$ of additive subgroups Λ_g of $(\Lambda, +)$ such that $\Lambda = \bigoplus_{g\in G} \in \Lambda_g$ and $\Lambda_g\Lambda_h \subseteq \Lambda_{gh}$ for every $g, h \in G$. The additive subgroup Λ_g called the homogeneous component of Λ. The set $H(\Lambda) = \cup_{g\in G}\Lambda_g$ is the set of homogeneous elements of Λ.
Let η be a right (resp. left) ideal of a graded ring Λ. Then η is said to be a graded right (resp. left) ideal if $\eta = \bigoplus_{g\in G} \eta_G$, where $\eta_g = (\eta \cap \Lambda_g)$ for all $g \in G$. That is, for $x \in \eta, x = \sum_{g\in G} x_g$, where $x_G \in \eta$ for all $g \in G$. A graded ring Λ is said to be gr-prime (gr-semiprime) if $a\Lambda b = \{0\}$ implies $a = 0$ or $b = 0$ (if $a\Lambda a = \{0\}$ then $a = 0$), where $a, b \in H(\Lambda)$. Moreover, a graded ring Λ is a gr-semiprime ring if the intersection of all the gr-prime ideals is zero.
Here, we establish interesting results related to homogeneous homoderivations. We prove the existence of a non-trivial family of homoderivations that are not homogeneous on graded rings. Furthermore, based on homogeneous homoderivations, we extend certain existing significant results in the context of prime (resp. semiprime) rings to gr-prime (resp. gr-semiprime) rings.

Theorem 1. *Let Λ be a gr-semiprime ring with a 2-torsion free property. If κ is a homogeneous homoderivation and $c \in H(\Lambda)$ such that $[c, \kappa(x)] \in Z(\Lambda)$ for all $x \in \Lambda$, then $\kappa = 0$ or $c \in Z(\Lambda)$.*

Theorem 2. *Let Λ be a 2-torsion free gr-semiprime ring with a 2-torsion free property and gr-prime ideal η be a gr-prime ideal Λ. Suppose κ_1 and κ_2 be homoderivations of Λ. Suppose that at least one of κ_1 and κ_2 is homogeneous and their composition $\kappa_1\kappa_2$ is a derivation. Then either $\kappa_1 \in \eta$ or $\kappa_2 \in \eta$.*

Proposition 3. *Let Λ be a gr-prime ring and η a non zero graded left ideal of Λ. If κ is a non zero homogeneous homoderivation of Λ, then its restriction on η is non zero.*

References

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